MICROECONOMICS COMPREHENSIVE EXAM

MAY 2016

Instructions:

(1) Please answer each of the four questions on **separate** pieces of paper.
(2) Please write only on **one side** of a sheet of paper
(3) Please write in **pen only**
(4) When finished, please arrange your answers **alphabetically** (in the order in which they appeared in the questions, i.e. 1 (a), 1 (b), etc.)

1. In a three-person, two-good economy, let x_i^j denote the consumption of good j ($j = 1, 2$) by consumer i ($i = 1, 2, 3$). Let the preferences of consumers be given as

$$u_i(x_1^1, x_2^1, x_i^2) = \frac{1}{2}\log x_1^1 + \frac{1}{2}\log x_2^1 + x_i^2, \text{ for } i = 1, 2,$$

and

$$u_3(x_3^1, x_3^2) = \log x_3^1 + x_3^2.$$

You may view agents 1 and 2 as friends (or neighbors), who derive as much utility from their own consumption of good 1 as from from their friend's consumption. Agent 3 is not a friend (or neighbor) of agents 1 and 2, so that he derives no utility from the other agents' consumption, nor does his consumption create any externalities for agents 1 and 2. Each consumer is initially endowed with one unit of each good.

(a) Carefully define a Walrasian equilibrium in this economy.

(b) Find all Walrasian equilibria for this economy. It is easier to do this if you normalize the prices so that good 2 is the numeraire commodity, and p is the price of good 1.

(c) Characterize Pareto optimal allocations for this economy as a programming problem. Check if the equilibrium allocation(s) is (are) Pareto optimal.

2. A risk-neutral investor contemplates investment of 1 unit of capital into a risky project that will generate a random value one period after the investment had been made. The value of the project at time $t = 0, 1, \dots$ is e^{X_t}, where $X_0 \equiv x$ is deterministic, and X_t is determined by the color of a ball drawn at random from an urn that contains either red (r), yellow (y) or green (g) balls. Namely, when a ball is drawn from the urn at time $t = 1$,

$$X_1 = \begin{cases} x+3, & \text{if g} \\ x+1, & \text{if r} \\ x-1, & \text{if y} \end{cases}.$$

For $t = 2, \dots$, $X_t = X_1$, i.e., all the uncertainty is resolved at $t = 1$.

Let $p_c \geq 0$ denote the probability of the event that a ball of color $c \in \{r, g, y\}$ is drawn; then $p_r + p_y + p_g = 1$. Given the current value of the project e^x, it is easy to see that the expected value of the project is $\phi(e)e^x$, where

$$\phi(e) = p_g e^3 + p_r e + p_y e^{-1}.$$

Parameters of the model (the distribution of colors in the urn) are specified as follows: for a given $p_g \in [0, 0.5/(e^2+1)]$, set

$$p_r = 0.5 - p_g(e^2+1), \ p_y = 0.5 + p_g e^2. \tag{1}$$

It follows from (1) that the urn contains balls of at least two different colors, one of which is yellow. If there are no green balls, then exactly half of the balls are yellow. If there are no red balls, then more than half of the balls are yellow.

Notice that the parameters of the model are such that for any $p_g \in [0, 0.5/(e^2+1)]$,

$$\phi(e) = \phi(e; p_g) = \phi(e; 0).$$

Assume, first, that p_g is known.

(a) Let $\beta \in (0, 1)$ be the discount factor per each time period. Argue that it is never optimal to invest at $t \geq 2$.

(b) Let $Y = e^x$. Let $V_0(Y)$ denote the net present value of investment made at $t = 0$. Recall that the investment has to generate a non-negative value, otherwise, it is better not to invest, and that the value accrues a period after the investment had been made. Therefore $V_0(Y) = \max\{\beta\phi(e)Y - 1, 0\}$, equivalently,

$$V_0(Y) = \begin{cases} \beta\phi(e)Y - 1, & \text{if } Y > \frac{1}{\beta\phi(e)} \\ 0, & \text{if } Y \leq \frac{1}{\beta\phi(e)} \end{cases}.$$

Let $V_1(Y)$ denote the net present value of investment made at $t = 1$ evaluated at $t = 0$. Then $V_1(Y) = \beta E_Y^{p_g}[\max\{\beta Y_1 - 1, 0\}]$, where $E_Y^{p_g}$ denotes the expectation operator parameterized by p_g and conditioned on the current state Y, and $Y_1 = e^{X_1}$. Write down the expression for $V_1(Y)$ similar to the above expression for $V_0(Y)$.

(c) The investor has to decide when (if ever) it is optimal to invest. We can write the investor's problem as follows: given Y,

$$\max_{t\in\{0,1\}} V_t(Y).$$

Show that there exists $Y^* = Y^*(p_g) \in \left(\frac{1}{\beta\phi(e)}, \frac{e}{\beta}\right)$ such that $V_0(Y^*) = V_1(Y^*)$. Furthermore, if $Y \geq Y^*$, then the investor invests into the project immediately, and if $Y < Y^*$, the investor waits until $t = 1$, and then invests only if $Y_1 > Y$; otherwise she never invests.

(d) Now assume that $p_g \in [0, 0.5/(e^2+1)]$ is not known. Let the investor be ambiguity averse so that her preferences are represented by maxi-min expected utility. We can write the investor's problem as follows: given Y,

$$\max_{t\in\{0,1\}} \min_{p_g\in[0,0.5/(e^2+1)]} V_t(Y).$$

Show that Y^* that makes the investor indifferent between immediate investment and delaying the decision to invest until $t = 1$ is calculated as if $p_g = 0$.

Hint: Use the notion of second order stochastic dominance and the fact $\max\{\beta e^x - 1, 0\}$ is a convex function.

3. (Based on Abreu (1986)) Consider the following oligopoly problem. There are n firms, indexed by i. In the stage game, each firm i chooses a quantity $a_i \in \mathbb{R}^+$. Given outputs, market price is given by $1 - \sum_{i=1}^n a_i$, when this number is nonnegative, and 0 otherwise. Each firm has a constant marginal cost of $c < 1$. The payoff of firm i from outputs $a_1, \ldots, a_n$ is then given by

$$u_i(a_1, \ldots, a_n) = a_i \Big(\max \Big\{ 1 - \sum_{j=1}^n a_j, 0 \Big\} - c \Big).$$

(a) Derive $(a^{NE}, \ldots, a^{NE})$, the pure strategy Nash equilibrium of the stage game. What are the individual payoffs in this Nash equilibrium?

(b) Let $(a^m, \ldots, a^m) \in (\mathbb{R}^+)^n$ denote the symmetric allocation that maximizes joint profits. Calculate a^m. What are the individual payoffs in this case?

(c) Define and calculate $\underline{v}_i$, firm i's minmax payoff.

This stage game is repeated infinitely many times. Each firm evaluates payoffs in the infinitely repeated game according to the average discounted criterion with discount factor $\delta \in (0, 1)$. We restrict attention to *strongly symmetric equilibria* of the repeated game: these are equilibria in which, after each history (including off-path histories), the same quantity is chosen by every firm. We distinguish the (potentially asymmetric) action profile $a \in (\mathbb{R}^+)^n$ from the (commonly chosen) output level $q \in \mathbb{R}^+$. Let $\mu(q)$ be the stage-game payoff obtained by each of the n firms when they each produce output level $q \in \mathbb{R}^+$:

$$\mu(q) = q(\max\{1 - nq, 0\} - c).$$

Let $\mu^d(q)$ be the payoff to a single firm when every other firm produces output q and the firm in question maximizes its stage-game payoff. Hence (exploiting the symmetry to focus on 1's payoff),

$$\mu^d(q) = \max_{a_1 \in \mathbb{R}^+} u_1(a_1, q, ..., q).$$

(d) Describe the functions $\mu(q)$ and $\mu^d(q)$. For what value of q, if any, do we have $\mu(q) = \mu^d(q)$? Sketch the functions $\mu(q)$ and $\mu^d(q)$, carefully placing the output levels a^{NE} and a^m in your figure.

(e) Derive δ^{GT}, the lower bound on the discount factor above which the collusive outcome can be supported in a subgame-perfect equilibrium by a Grim Trigger strategy reverting forever to the stage-game Nash equilibrium after any output other than $(a^m, a^m, \ldots, a^m)$.

For any two quantities of output $(\bar{q}, \underline{q})$, define a *carrot-and-stick* punishment, $\sigma(\bar{q}, \underline{q})$, to be strategies in which all firms play $\underline{q}$ in the first period and thereafter play $\bar{q}$, with any deviation from these strategies causing this prescription to be repeated. Intuitively, $\underline{q}$ is the "stick" and $\bar{q}$ the "carrot." The punishment specifies a single-period penalty followed by repeated play of the carrot. Deviations from the punishment simply cause it to begin again. We restrict attention to output levels $q \geq a^m$.

(f) Show that the carrot-and-stick profile, $\sigma(\bar{q}, \underline{q})$, is a SPE if and only if:

$$\mu^d(\underline{q}) \leq (1-\delta)\mu(\underline{q}) + \delta\mu(\bar{q}), \quad \text{and} \quad \mu^d(\bar{q}) \leq \mu(\bar{q}) + \delta\big[\mu(\bar{q}) - \mu(\underline{q})\big].$$

(g) Can a firm be held to its min-max payoff by a carrot-and-stick punishment? What must the value of $\underline{q}$ be in this case? Give an economic interpretation for your findings. How do you expect δ^{GT} compares with δ^{CS}, the lower bound on the discount factor above which the collusive outcome can be supported in a subgame-perfect equilibrium by a carrot-and-stick punishment.

4. Consider a two-bidder, independent private-value auction with valuations uniformly distributed on $[0, 1]$. Compare the following assumptions on utilities: (a) bidder i ($i \in \{1, 2\}$) has utility $v_i - P$ when she wins the object and has to pay P, while her outside option is normalized to zero; (b) bidder i ($i \in \{1, 2\}$) has utility $\sqrt{v_i - P}$ when she wins the object and has to pay P, while her outside option is normalized to zero.

(a) Compare the seller's expected revenue in cases (a) and (b) for the second-price, sealed-bid auction.

(b) Compare the seller's expected revenue in cases (a) and (b) for the first-price, sealed-bid auction. (Hint: look for a symmetric equilibrium where bids are increasing in the bidder's valuation.)

(c) Discuss your findings, particularly in light of the revenue equivalence theorem.