

Credible Mechanism Design*

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Abstract

Consider an extensive-form mechanism, run by an auctioneer who communicates sequentially and privately with agents. Suppose the auctioneer can make any deviation that no single agent can detect. We study the mechanisms such that it is incentive-compatible for the auctioneer not to deviate – the *credible* mechanisms. Consider the *ex post* individually-rational optimal auctions. The first-price auction is the unique sealed-bid credible mechanism. The ascending auction is the unique strategy-proof credible mechanism.

Keywords: Mechanism design, auction, credible, strategy-proof, sealed-bid.

JEL classification codes: D44, D47, D82

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1 Introduction

Auctions are used to sell a wide variety of goods - art, fish, real estate, treasury bonds, internet advertising, wireless spectrum, and, in 193 AD, the entire Roman Empire (Shubik, 2004). The theory of optimal auctions pins down only a few features of the auction format. Under the standard assumptions, if bidders with higher values submit higher bids, and the reserve price is optimally chosen, then the auction is optimal (Myerson, 1981). Despite this, most real-world auctions are variations on just a few classic formats - the first-price auction, the ascending auction, and (more recently) the second-price auction (Cassady, 1967; McAfee and McMillan, 1987).¹ Why?

In this paper, we shed light on this question by proposing a new framework for mechanism design under partial commitment. The standard mechanism design paradigm assumes full commitment on behalf of the auctioneer:

[In the standard paradigm] it is assumed that the organizer of the auction has the ability to commit himself in advance to a set of policies. He binds himself in such a way that all the bidders know that he cannot change his procedures after observing the bids, even though it might be in his interest ex post to renege. In other words, the organizer of the auction acts as the Stackelberg leader or first mover. (McAfee and McMillan, 1987)

In real-world auctions, however, the auctioneer may have various opportunities to deviate that are difficult for bidders to detect. The auctioneer could secretly inspect and tamper with sealed bids. She could hire confederates to pose as bidders to manipulate the outcome. She could call out nonexistent bids in order to give the impression of greater demand - a practice known as “chandelier bidding”:

Under New York City regulations auctioneers can fabricate bids up to an item’s reserve price. Because a reserve price is secret and not listed in the catalog, bidders have no way of knowing which offers are real. (The New York Times, April 24 2000)

Which deviations are feasible plainly depends on the auction format. Instead of considering deviations in an *ad hoc* manner, it would be useful to model the feasible deviations as a function of the format. This paper provides one systematic framework to do so.

Consider any *protocol*; a tuple consisting of an extensive-form mechanism and a strategy profile for the agents. We can think of the auctioneer *running* the mechanism by engaging in a sequence of private communication with the agents. Upon encountering an information set, she picks up the telephone and conveys that information to the agent who

¹The Dutch (descending) auction, in which the price falls until one bidder claims the object, is less prevalent (Krishna, 2010, p.2).

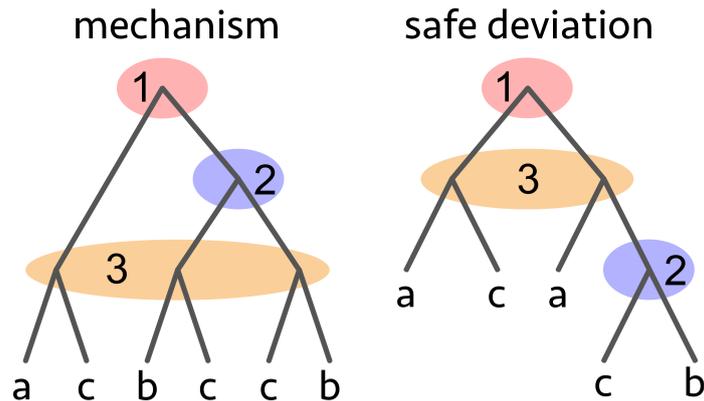


Figure 1: A mechanism and a deviation. If agent 1 cannot distinguish outcomes a and b , then the deviation is safe.

is called to play, along with a set of acceptable replies (actions). The agent then chooses a reply. The auctioneer keeps making telephone calls, sending information and receiving replies, until she reaches a terminal history, whereupon she chooses the corresponding outcome and the game ends.

Suppose some utility function for the auctioneer. For instance, assume that the auctioneer wants revenue. Suppose that each agent intrinsically observes certain features of the outcome. For instance, each agent observes whether or not he wins the object, and how much he pays, but not how much other agents pay.

After playing in the mechanism, each agent observes a sequence of communication between himself and the auctioneer and some features of the outcome. Even if the auctioneer deviates from her assigned strategy, agent i 's observation could still have an *innocent explanation*. That is, when the auctioneer plays by the rules, there exist types for the other agents that result in that same observation for i .

For any given protocol, there may be some deviations that are *safe*, in the sense that for every type profile, each agent's observation has an innocent explanation. That is, every observation that an agent might have (under the deviation) is also an observation he might have when the auctioneer is running the mechanism. For instance, when a bidder bids \$100 in a second-price auction, receives the object, and is charged \$99, that observation has an innocent explanation - it could be that the second-highest value was \$99. Thus, in a second-price auction, the auctioneer can safely deviate by exaggerating the second-highest bid.

Instead of just choosing a different outcome, the auctioneer may also alter the way she communicates with agents. For example, consider the mechanism on the left side of Figure 1. Each agent has one information set, two moves (left and right), and two types (θ_i^{left} and θ_i^{right}) that play the corresponding moves. By assumption, agent 1 observes whether the outcome is in the set $\{a, b\}$ or in $\{c\}$. Agents 2 and 3 perfectly observe the outcome.

The right side of Figure 1 illustrates a safe deviation: If agent 1 plays left, then the auctioneer plays ‘by the book’. If agent 1 plays right, then instead of querying agent 2, the auctioneer queries agent 3. If agent 3 then plays left, the auctioneer chooses outcome a . If agent 3 plays right, only then does the auctioneer query agent 2, choosing c if 2 plays left and b if 2 plays right.

For every type profile, each agent’s observation has an innocent explanation. The most interesting case is when the type profile is $(\theta_1^{\text{right}}, \theta_2^{\text{left}}, \theta_3^{\text{left}})$. In this case, playing by the book results in outcome b , but the deviation results in outcome a . Agent 1 cannot distinguish between a and b , so $(\theta_2^{\text{left}}, \theta_3^{\text{left}})$ is an innocent explanation for 1. $(\theta_1^{\text{left}}, \theta_3^{\text{left}})$ is an innocent explanation for 2, and $(\theta_1^{\text{left}}, \theta_2^{\text{left}})$ is an innocent explanation for 3.

Notably, this deviation involves not just choosing different outcomes, but communicating differently even before a terminal history is reached. Indeed, when the type profile is $(\theta_1^{\text{right}}, \theta_2^{\text{left}}, \theta_3^{\text{left}})$, the auctioneer can only get outcome a by deviating midway. If she waited until the end and then deviated to choose a , then agent 2’s observation would not have an innocent explanation. Once agent 2 is called to play, he knows that outcome a should not occur.

A protocol is *credible* if running the mechanism is incentive-compatible for the auctioneer; that is, if the auctioneer prefers playing by the book to any safe deviation. This is a way to think about partial commitment power for any extensive-form mechanism. For instance, if the auctioneer wants revenue, then the second-price auction is not credible. Similarly, in Figure 1, if the auctioneer prefers outcome a to any other outcome, then the mechanism is not credible.

Our model assumes that the auctioneer is the nexus of communication, exchanging private messages with individual agents. Sometimes, this is literally the case: Large auction houses, such as Sotheby’s and Christie’s, convey information and solicit bids over multiple telephone lines and the Internet; most of Christie’s auctions are won by bidders who are not in the room (Grant, 2014). Sometimes, the law forbids bidders from communicating with each other, as in the 2017 US Federal Communications Commission Incentive Auction for wireless spectrum.² Of course, many auctions involve other communication procedures.³ We see our definition of a “safe deviation” as a tractable shorthand for a rich set of real-world deviations, such as tampering with sealed bids, altering the relative speed of price clocks, or hiring confederates to influence the outcome.

Having defined the framework, we now turn to our main application. Consider three canonical auction formats. The first-price auction is sealed-bid - each agent is called to play exactly once, and has no information about the history of play when selecting his

²The Incentive Auction’s rules state that bidders “are prohibited from communicating directly or indirectly any incentive auction applicant’s bids or bidding strategies” (Section 1.2205(b)).

³Cryptographic techniques could allow bidders to allocate the object by communicating directly with each other, rather than via a centralized auctioneer (Bogetoft et al., 2009). These techniques are a useful alternative to standard auctions, but may be costly to set up and explain to bidders.

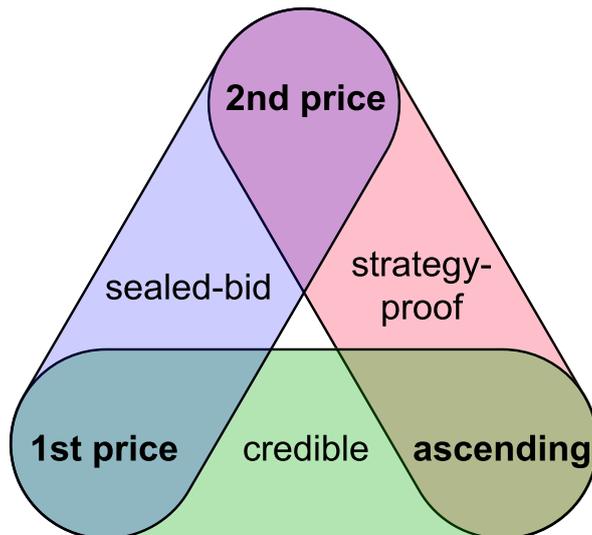


Figure 2: An auction trilemma: In the class of *ex post* individually-rational optimal auctions, no auction is sealed-bid, strategy-proof, and credible. Picking two out of three desirable properties uniquely characterizes each classic format.

action. This yields a substantial advantage: Sealed-bid auctions can be conducted rapidly and asynchronously, thus saving logistical costs.⁴ The ascending auction is strategy-proof. Thus, it demands less strategic sophistication from bidders, and does not depend sensitively on bidders' beliefs (Wilson, 1987; Bergemann and Morris, 2005; Chung and Ely, 2007). The second-price auction is sealed-bid and strategy-proof; it combines the virtues of the first-price auction and the ascending auction (Vickrey, 1961). However, many real-world auctioneers persist in running first-price auctions and ascending auctions, despite the invention of this (apparently) superior format (Rothkopf et al., 1990). Why is this so?

Credibility illuminates the relationship between these three canonical auctions. Suppose the classic assumptions: Private values, with symmetric independent regular distributions (Myerson, 1981). We restrict attention to *ex post* individually rational auctions, so only the winner makes payments. The second-price auction is the unique sealed-bid strategy-proof optimal auction, by the Green-Laffont-Holmström theorem (Green and Laffont, 1977; Holmström, 1979).

The results that follow require us to bridge the discrete world of extensive game forms and the continuous world of optimal auctions. We suppress these technicalities in the introduction, but the reader should be aware that these results hold ‘in the limit’, as a finite grid type space becomes arbitrarily fine.

Our first result is as follows: The first-price auction (with an optimal reserve) is the unique credible sealed-bid optimal auction. This implies that, in the class of sealed-bid

⁴Using data from U.S. Forest Service timber auctions, Athey et al. (2011) find that “sealed bid auctions attract more small bidders, shift the allocation toward these bidders, and can also generate higher revenue”.

mechanisms, there is a sharp trade-off between incentive-compatibility for the auctioneer and dominant strategies for the agents.

Sealed-bid mechanisms include the direct revelation mechanisms, in which each agent simply reports his type. Thus, when designing credible protocols, restricting attention to revelation mechanisms loses generality. The problem is that revelation mechanisms reveal too much, too soon. For an agent to have a dominant strategy, his payment must depend on the other agents' types. If the auctioneer knows the entire type profile, and the winning bidder's payment depends on the other bidders' types, then the auctioneer can safely deviate to raise revenue. This makes it impossible to run a credible strategy-proof optimal auction. What happens when we look outside the class of revelation mechanisms - when we use the full richness of extensive forms to regulate who knows what, and when?

Our second result is as follows: The ascending auction (with an optimal reserve) is credible. Moreover, it is the *unique* credible strategy-proof optimal auction. No other extensive forms satisfy these criteria.

These results imply an auction trilemma. Sealed bids, strategy-proofness, and credibility are all desirable properties. An optimal auction can have any two of these properties, but not all three at once. Moreover, picking two out of three characterizes each of the canonical auction formats (first-price, second-price, and ascending). Figure 2 illustrates.

1.1 Related work

In the literature on mechanism design with an informed principal, the principal has exogenous private information and cannot commit to how she uses it. Thus, the mechanism must give the principal incentives to report that information to a neutral mediator (Myerson, 1983; Maskin and Tirole, 1990, 1992). In this paper, we study a case of endogenous private information. Our principal is the mediator - she learns, in the course of running the mechanism, private information about each agent's type, and needs incentives to follow the rules.

There has been substantial prior interest in auction design under limited commitment. The main approaches are as follows: Some papers consider models in which an auctioneer runs auctions over multiple periods, but cannot commit today to the mechanism used tomorrow (Milgrom, 1987; McAfee and Vincent, 1997; Caillaud and Mezzetti, 2004; Bester and Strausz, 2001; Liu et al., 2014; Skreta, 2015). (This approach often relies on equilibrium refinements to yield interesting results.) Other papers consider bargaining games in which the auctioneer cannot commit to end the auction - she can continue to make or solicit offers in defiance of the rules (Burguet and Sakovics, 1996; McAdams and Schwarz, 2007; Vartiainen, 2013; Lobel and Paes Leme, 2017). Some papers consider mechanisms in which each agent reports his type to the principal, and the principal's contract with each agent can only directly depend on that bidder's report (Rothkopf and Harstad, 1995;

Porter and Shoham, 2005; Dequiedt and Martimort, 2015). Finally, Loertscher and Marx (2017) consider a clock auction in which the auctioneer chooses whether to keep the clocks running or to end the auction.

Our current approach is distinct in several ways. Firstly, we consider a non-repeated interaction, and thus do not rely on equilibrium refinements. Secondly, our auctioneer’s commitment power is limited by what agents can observe, not by the passage of calendar time. We thus rule out flagrant rule-breaking - for instance, the auctioneer cannot make new offers to a bidder who knows that the auction should already be over. Thirdly, instead of assuming that each bidder makes a single report, or assuming that every bidder makes one offer in every period, we analyze partial commitment for the class of extensive game forms with perfect recall. We show that credibility is a defining characteristic of the first-price auction and the ascending auction. To the best of our knowledge, no prior work has produced a desirable criterion that selects these two venerable auction formats.

Li (2017) introduces a notion of bilateral commitment power, to characterize the choice rules that are implementable in obviously dominant strategies. Our paper builds on that formalism, with two key differences. Firstly, the notion of partial commitment in Li (2017) is only for dominant-strategy mechanisms, whereas credibility allows for Bayes-Nash mechanisms. Secondly, obvious dominance is a stronger incentive requirement for the agents, but is silent on the incentives of the auctioneer. Credibility imposes no novel restrictions on the incentives of agents. Instead, it explicitly requires the mechanism to be incentive-compatible for the auctioneer.

2 The Model

2.1 Definitions

The environment consists of:

1. A finite set of agents, N .
2. A set of outcomes, X .
3. A finite type space, $\Theta_N = \times_{i \in N} \Theta_i$.
4. A joint probability distribution $D : \Theta_N \rightarrow [0, 1]$.
5. Agent utilities $u_i : X \times \Theta_N \rightarrow \mathbb{R}$
6. A partition Ω_i of X for each $i \in N$. (ω_i denotes a cell of Ω_i .)

The partition Ω_i represents what agent i *intrinsically observes* about the outcome. Conceptually, these partitions represent physical facts about the world, which are not

objects of design. They capture the bare minimum that each agent observes about the outcome, regardless of the choice of mechanism.⁵

A mechanism is an extensive game form with consequences in X . This is an extensive game form for which each terminal history is associated with some outcome. Formally, a mechanism G is a tuple $(H, \prec, P, A, \mathcal{A}, (\mathcal{I}_i)_{i \in N}, g)$, where each part of the tuple is as specified in Table 1. The full definition of extensive forms is familiar to most readers, so we relegate further detail to Appendix A. Let \mathcal{G} denote the set of all extensive game forms with consequences in X with finitely many histories and perfect recall.

Table 1: Notation for Extensive Game Forms

Name	Notation	Representative element
histories	H	h
precedence relation over histories	\prec	
initial history	h_\emptyset	
terminal histories	Z	z
player called to play at h	$P(h)$	
actions	A	a
most recent action at h	$\mathcal{A}(h)$	
information sets for agent i	\mathcal{I}_i	I_i
outcome resulting from z	$g(z)$	
immediate successors of h	$\sigma(h)$	
actions available at I_i	$A(I_i)$	

S_i denotes a (pure) strategy: For each information set where agent i is called to play and each type of i , S_i chooses an action $S_i(I_i, \theta_i) \in A(I_i)$. $(S_i)_{i \in N} \equiv S_N$ denotes a strategy profile for the agents, for some $G \in \mathcal{G}$. We refer to a pair (G, S_N) as a *protocol*. Let $x^G(S_N, \theta_N)$ denote the outcome in G , when agents play according to S_N and the type profile is θ_N . Let $u_i^G(S_N, \theta_N) \equiv u_i(x^G(S_N, \theta_N), \theta_N)$.

Definition 1. (G, S_N) is **Bayesian incentive-compatible (BIC)** if, for all $i \in N$,

$$S_i \in \operatorname{argmax}_{S'_i} \mathbb{E}_{\theta_N} [u_i^G(S'_i, S_{-i}, \theta_N)] \quad (1)$$

We explicitly model the mechanism designer (the *auctioneer*) as a player⁶. In particular, we consider:

1. An auctioneer, denoted 0.
2. Auctioneer utility $u_0 : X \times \Theta_N \rightarrow \mathbb{R}$

⁵In the application that follows, we will assume that each bidder in an auction knows how much he paid and whether he receives the object. In effect, this rules out the possibility that the auctioneer could hire pickpockets to raise revenue, or sell the object to multiple bidders by producing counterfeit copies.

⁶We use the term ‘auctioneer’ to refer to the mechanism designer, but readers should note that this could be any mediator who runs a mechanism, such as a school choice authority or the National Resident Matching Program.

2.2 Pruning

We restrict attention to the class of *pruned* protocols.⁷ This technique allows us to remove redundant parts of the game tree, and implies cleaner definitions for the theorems that follow. In words, a pruned protocol has three properties.

1. For every history h , there exists some type profile such that h is on the path of play.
2. At every information set, there are at least two actions available (equivalently, every non-terminal history has at least two immediate successors).
3. If agent i is called to play at history h , then there are two types of i compatible with his actions so far, that could lead to different eventual outcomes.

Let $z^G(S_N, \theta_N)$ denote the terminal history that results from (S_N, θ_N) . Formally,

Definition 2. (G, S_N) is **pruned** if, for any history h :

1. There exists θ_N such that $h \preceq z^G(S_N, \theta_N)$
2. If $h \notin Z$, then $|\sigma(h)| \geq 2$.
3. If $h \notin Z$, then for $i = P(h)$, there exist $\theta_i, \theta'_i, \theta_{-i}$ such that
 - (a) $h \prec z^G(S_N, (\theta_i, \theta_{-i}))$
 - (b) $h \prec z^G(S_N, (\theta'_i, \theta_{-i}))$
 - (c) $x^G(S_N, (\theta_i, \theta_{-i})) \neq x^G(S_N, (\theta'_i, \theta_{-i}))$

By the next proposition, when our concern is BIC implementation, it is without loss of generality to consider only pruned mechanisms.

Proposition 1. *If (G, S_N) is BIC, then there exists (G', S'_N) such that (G', S'_N) is pruned and BIC and for all θ_N , $x^{G'}(S'_N, \theta_N) = x^G(S_N, \theta_N)$.*

Hence, from this point onwards we restrict attention to pruned (G, S_N) .

2.3 A partial commitment game

Consider the messaging game, defined thusly:

1. The auctioneer chooses to:
 - (a) Either: Select $x \in X$ and end the game.
 - (b) Or: Go to step 2.

⁷This is stronger than the notion of pruning used in Li (2017), which includes only the first requirement.

2. The auctioneer chooses some agent $i \in N$ and sends a message along with a set of acceptable replies (m, R) .
3. Agent i privately observes (m, R) and chooses $r \in R$.
4. The auctioneer privately observes r .
5. Go to step 1.

Assume the auctioneer has an arbitrarily rich message space M . At any prior round k , the auctioneer messaged $i_k \in N$ with (m^k, R^k) , and received reply $r^k \in R_k$.

Let \mathcal{S}_0 denote the set of auctioneer pure strategies; we restrict these to be of finite length. A strategy for the auctioneer specifies what to do next, as a function of the entire history of communications: $S_0((i^k, m^k, R^k, r^k)_{k=1}^t) \in (N \times M \times (2^M \setminus \{\emptyset\})) \cup X$.

A strategy for agent i specifies what reply to give, as a function of the previous communications between agent i and the auctioneer, and the agent's type. Let m_i^k denote the k th message that the auctioneer sent to agent i , and similarly for R_i^k and r_i^k . Let t_i denote the total number of messages sent to agent i so far. $S_i((m_i^k, R_i^k, r_i^k)_{k=1}^{t_i-1}, m^{t_i}, R^{t_i}, \theta_i) \in R^{t_i}$.⁸

For any $S_0 \in \mathcal{S}_0$ and any S_N , (S_0, S_N) results in some sequence of communication between the auctioneer and agent i , $o_i^c \equiv (m_i^k, R_i^k, r_i^k)_{k=1}^{T_i}$ and some outcome x . Let o_i^x denote $\omega_i \in \Omega_i \mid x \in \omega_i$. $o_i = (o_i^c, o_i^x)$ denotes an *observation* for agent i , and O_i denotes the set of all possible observations for agent i . $\phi_i(S_0, S_N, \theta_N)$ denotes the unique observation resulting from (S_0, S_N) , when the type profile is θ_N .

Definition 3. Take any $G = \langle H, \prec, P, A, \mathcal{A}, (\mathcal{I}_i)_{i \in N}, g \rangle$. S_0 **runs** G if there exists a one-to-one function $\lambda : (\mathcal{I}_i)_{i \in N} \cup A \rightarrow M$ such that S_0 is described by the following algorithm, where we initialize $h := h_\emptyset$.

1. If $h \in Z$, terminate and select $x = g(h)$.
2. Else:
 - (a) Choose agent $P(h)$ and send $(m, R) = \lambda(I_i, A(I_i))$ for I_i such that $h \in I_i$.
 - (b) Upon receiving $r \in R$, choose h' such that $\mathcal{A}(h') = \lambda^{-1}(r)$ and $h' \in \sigma(h)$. Set $h := h'$ and go to step 1.

We use S_0^G to denote an auctioneer strategy that runs G .

Given S_0^G , for any S_i , we can define an equivalent strategy \tilde{S}_i for agent i in the partial commitment game. Given any (m, R) , \tilde{S}_i selects reply $\lambda(S_i(\lambda^{-1}(m)))$.⁹ We abuse notation and use S_i to denote both a strategy for agent i in G , and the equivalent strategy for agent i in the partial commitment game.

⁸Note the lack of calendar time: The agent observes the sequence of past communications between himself and the auctioneer, *not* a sequence of periods in which he either sees some communication or none.

⁹We specify \tilde{S}_i arbitrarily for communication sequences that are never observed under S_0 .

2.4 Credible mechanisms

We now define the required machinery to introduce feasible deviations for the auctioneer. First, we say that some observation o_i has an “innocent explanation” (with respect to S_0) if there exist some types of other agents such that if they had those types and the auctioneer played S_0 , o_i was the observation of agent i .

Definition 4. *Suppose the auctioneer promises to play S_0 and the type profile is θ_N . i 's observation o_i has an **innocent explanation** if there exists θ'_{-i} such that $o_i = \phi_i(S_0, S_N, (\theta_i, \theta'_{-i}))$.*

Definition 5. *Suppose the auctioneer promises to play S_0 . Then, an auctioneer strategy $S'_0 \in \mathcal{S}_0$ is **safe** if for all players $i \in N$ and all type profiles $\theta_N \in \Theta_N$, $\phi_i(S'_0, S_N, \theta_N)$ has an innocent explanation.*

Let $\mathcal{S}_0^*(S_0) \equiv \{S'_0 \mid S'_0 \text{ is safe given promise } S_0\}$ be the set of all safe strategies of the auctioneer. The function $\mathcal{S}_0^*(\cdot)$ takes an auctioneer strategy as input and generates all possible strategies that the auctioneer can deviate to without being detected by a single player.

Let $u_0(S_0, S_N, \theta_N)$ denote the auctioneer's utility in the partial commitment game, when the strategy profile is (S_0, S_N) and the type profile is θ_N .

Definition 6. (G, S_N) is **credible** if:

$$S_0^G \in \operatorname{argmax}_{S_0 \in \mathcal{S}_0^*(S_0^G)} \mathbb{E}_{\theta_N} [u_0(S_0, S_N, \theta_N)] \quad (2)$$

A protocol is credible if playing ‘by the book’ maximizes the auctioneer's expected utility.

Definition 6 takes the expectation of θ_N with respect to the joint distribution D , but it implicitly requires the auctioneer to best-respond to her *updated* beliefs in the course of running G . Recall that a strategy for the auctioneer is a complete contingent plan. Suppose that in the course of running G , the auctioneer discovers new information about agents' types, such that she can profitably change her continuation strategy. There exists a deviating strategy that adopts this new course of action contingent on the auctioneer discovering this information, and plays by the book otherwise.

We can extend our pruning proposition to credible implementation as well.

Proposition 2. *If (G, S_N) is credible and BIC, then there exists (G', S'_N) such that (G', S'_N) is pruned, credible, and BIC, and for all θ_N , $x^{G'}(S'_N, \theta_N) = x^G(S_N, \theta_N)$.*

By Proposition 2, when our concern is credible implementation, it is without loss of generality to consider only pruned mechanisms.

Observation 1. (G, S_N) is credible and BIC if and only if (S_0^G, S_N) is a Bayes-Nash equilibrium of the partial commitment game in which the auctioneer is constrained to play strategies in $\mathcal{S}_0^*(S_0^G)$.

Credibility restricts attention to ‘promise-keeping’ equilibria of the partial commitment game. This is without loss of generality, in the sense that any equilibrium can be turned into a promise-keeping equilibrium by altering the promise.

Observation 2. If $S'_0 \in \mathcal{S}_0^*(S_0)$, then $\mathcal{S}_0^*(S'_0) \subseteq \mathcal{S}_0^*(S_0)$. Thus, if (S'_0, S_N) is a Bayes-Nash equilibrium given promise S_0 , then (S'_0, S_N) is a Bayes-Nash equilibrium given promise S'_0 .

Credibility is neither weaker nor stronger than the definition of bilateral commitment studied in Li (2017). That definition essentially requires that each agent’s strategy S_i is weakly dominant, even when the auctioneer can deviate to strategies that produce identical observations for i . By contrast, credibility requires S_0^G to be incentive-compatible for the auctioneer, when the auctioneer can make any safe deviation.

3 Credible Optimal Auctions

We consider an auction for a single object. We study the independent private values model of Myerson (1981), but with discrete type spaces. The use of discrete type spaces allows us to bypass the known paradoxes of extensive game forms with continuous time or infinite actions (Simon and Stinchcombe, 1989; Myerson and Reny, 2016). We restrict attention to *ex post* individually rational mechanisms, so only winning bidders make payments. An outcome $x = (y, t)$ consists of a winner $y \in N \cup \{0\}$ and a payment $t \in \mathbb{R}$ by the winner, so $X = (N \cup \{0\}) \times \mathbb{R}$. Assume there are at least two bidders.

Type spaces are discrete: $\Theta_i = \{\theta_i^0, \theta_i^1, \dots, \theta_i^K\}$. We associate each type with a real number, $v(\theta_i^k)$. For all i, j, k , $v(\theta_i^k) = v(\theta_j^k)$, and $v(\theta_i^{k+1}) - v(\theta_i^k) = \epsilon$ for all k . We will abuse notation slightly, and use θ_i^k to refer both to i ’s k th type, and to the real number associated with that type. It will be clear from context which is intended. Types are independently and identically distributed, with probability mass function $p : \Theta_i \rightarrow [0, 1]$. Every type has positive probability. Let $F(\theta^k) \equiv \sum_{l=0}^k p(\theta^l)$ and $f(\theta^k) \equiv \frac{p(\theta^k)}{\epsilon}$.

Agents have private values, that is:

$$u_i((y, t), \theta_N) = 1_{i=y}(\theta_i - t) \tag{3}$$

Ω_i is as follows: Each bidder observes whether he wins the object and observes his own payment. That is, $(y, t), (y', t') \in \omega_i$ if and only if:

1. $y \neq i$ and $y' \neq i$
2. or $y = y' = i$ and $t = t'$.

The auctioneer desires revenue¹⁰:

$$u_0((y, t), \theta_N) = 1_{y \in N} t \quad (4)$$

Let $\pi(G, S_N)$ denote the expected revenue of (G, S_N) .

The virtual values machinery in Myerson (1981) applies *mutatis mutandis* to the discrete setting. Suppose we choose (G, S_N) to maximize expected revenue subject to (interim) individual rationality and incentive compatibility. That is $\max_{G, S_N} \pi(G, S_N)$ subject to:

1. (G, S_N) is BIC.
2. For all i , θ_i , $\mathbf{E}_{\theta_{-i}}[u_i^G(S_N, \theta_N) \mid \theta_i] \geq 0$.

We $\check{u}_i^{G, S_N}(k, k')$ to denote the expected utility of agent i in G when his type is θ_i^k and he plays as though his type is $\theta_i^{k'}$. In order for a protocol to be optimal, certain individual rationality (IR) and incentive compatibility (IC) constraints must bind.

Proposition 3. (*Elkind, 2007*) *If (G, S_N) maximizes expected revenue, then:*

1. *IR-0 binds:* $\forall i : \check{u}_i^{G, S_N}(0, 0) = 0$
2. *IC binds locally downward:* $\forall i : \forall k \geq 1 : \check{u}_i^{G, S_N}(k, k) = \check{u}_i^{G, S_N}(k, k - 1)$

Proposition 4. *If IR-0 binds and IC binds locally downward, then:*

$$\pi(G, S_N) = \mathbf{E}_{\theta_N} \left[\sum_{i \in N} y_i^{G, S_N}(\theta_N) \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \right] \quad (5)$$

where $y_i^{G, S_N}(\theta_N)$ is an indicator equal to 1 if i wins the object when the type profile is θ_N (under (G, S_N)), and 0 otherwise.

We use $\eta(\theta_i) \equiv \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$ to denote the **virtual value** of type θ_i . By Proposition 4, if IR-0 binds and IC binds locally downward, then $\pi(G, S_N)$ is equal to the expected virtual value of the winning bidder.

We now define a **discrete** analogue to continuous optimality ('d-optimal').

Definition 7. (G, S_N) is **d-optimal** if (G, S_N) is BIC, IR-0 binds and y_i^{G, S_N} maximizes $\sum_{i \in N} y_i^{G, S_N}(\theta_N) \eta(\theta_i)$ subject to:

1. *Individual feasibility:* $\forall i : \forall \theta_N : y_i^{G, S_N}(\theta_N) \in \{0, 1\}$,
2. *Group feasibility:* $\forall \theta_N : \sum_i y_i^{G, S_N}(\theta_N) \leq 1$,

¹⁰The results that follow would require only small modifications if the auctioneer's payoff was a convex combination of revenue and social welfare.

3. *Monotonicity*: $\forall i : \forall \theta_i < \theta'_i : \mathbf{E}_{\theta_{-i}} \left[y_i^{G, S_N}(\theta_i, \theta_{-i}) \right] \leq \mathbf{E}_{\theta_{-i}} \left[y_i^{G, S_N}(\theta'_i, \theta_{-i}) \right]$.

d-optimality is a necessary condition for optimality (optimality would additionally require that IC binds locally downward). It is sufficient ‘in the limit’, in the sense that the expected revenue of any d-optimal mechanism is within ϵ of the optimal revenue. More generally, the expected revenue of any BIC mechanism is within ϵ of the expected virtual value of the winning bidder.

Proposition 5. *If (G, S_N) is BIC and IR-0 binds, then:*

$$\pi(G, S_N) \geq \mathbf{E}_{\theta_N} \left[\sum_{i \in N} y_i^{G, S_N}(\theta_N) \eta(\theta_i) \right] - \epsilon \quad (6)$$

Our results focus on d-optimality because the Revenue Equivalence Theorem does not hold for discrete type spaces. For instance, in a second-price auction with random tie-breaking, the local IC constraints do not bind: For θ_i strictly above the optimal reserve, agents with type θ_i strictly prefer bidding θ_i to bidding $\theta_i - \epsilon$. Thus, second-price auctions with random tie-breaking are not optimal, though they can be d-optimal.

Definition 8. *F is **regular** if $\eta(\theta_i)$ is strictly increasing in θ_i .*

Observation 3. *If F is regular, then (G, S_N) is d-optimal if and only if (G, S_N) is BIC, IR-0 binds, and for all θ_N :*

1. *If $\max_i \eta(\theta_i) > 0$, then $\sum_{i \in \arg \max \eta(\theta_i)} y_i^{G, S_N}(\theta_N) = 1$,*
2. *If $\eta(\theta_i) < 0$, then $y_i^{G, S_N}(\theta_N) = 0$.*

For regular F , let $\rho^* \in \Theta_i$ denote a **d-optimal reserve price**, with the property that if $\eta(\theta^k) < 0$ then $\theta^k < \rho$ and if $\eta(\theta^k) > 0$ then $\theta^k \geq \rho$.

Definition 9. *(G, S_N) **breaks ties in order** if there exists a strict total order \triangleright on N such that, if $\theta_i = \theta_j$ and $i \triangleright j$, then j does not win the object.*

Given (G, S_N) that breaks ties in order, we define a strict total order on all agent types, as follows: $\theta_i \triangleright \theta_j$ if and only if $\theta_i \geq \theta_j$ and either $\theta_i > \theta_j$ or $i \triangleright j$. We also include a reserve ρ in this total order: $\theta_i \triangleright \rho$ if and only if $\theta_i \geq \rho$. We use \min^\triangleright to denote the minimum of a set with respect to this \triangleright , and \max^\triangleright similarly.

3.1 Credible and sealed-bid optimal auctions

We now characterize credible and sealed-bid d-optimal auctions.

Definition 10. (G, S_N) is **sealed-bid** if every agent has exactly one information set and for every terminal history z , there exists $h \prec z$ such that $P(h) = i$.

(G, S_N) is an **almost-first-price auction** if (G, S_N) is sealed-bid, and for every agent, there exists $b_i : A(I_i) \rightarrow \mathbb{R}$ and a reserve ρ such that, after action profile $(a_i)_{i \in N}$:

1. Some bidder wins the object if and only if $\max_i b_i(a_i) \geq \rho$.
2. If i wins the object, then i pays $b_i(a_i)$.
3. If i wins the object then:
 - (a) Either: $i \in \operatorname{argmax}_j b_j(a_j)$.
 - (b) Or: $b_i(a_i) = \max_{a'_i} b_i(a'_i)$ and for all $j \neq i$, $i \triangleright j$.

The last condition deals with an unusual case: If the highest-priority bidder places his highest possible bid, then he wins for sure, even if that bid is slightly too low. For instance, suppose that $\Theta_1 = \Theta_2 = \{0, 5, 10\}$ and the reserve is 5. In an almost-first-price-auction, it could be that bidder 1's feasible bids are $\{0, 5, 8\}$ and bidder 2's feasible bids are $\{0, 5, 9\}$, but bidder 1 wins for sure if he bids 8. However, in any d-optimal equilibrium of the almost-first-price auction, this bid must be *almost* the highest bid. Suppose i wins all ties; i 's bid when his type is θ_i^K is weakly more than his bid when his type is $\theta_i^K - \epsilon$ (by incentive compatibility) which in turn is weakly more than j 's bid when j 's type is $\theta_j^K - \epsilon$ (by definition). Clearly, this anomaly vanishes for small ϵ .

Now we can state the main result of this section:

Theorem 1. Assume F is regular and (G, S_N) is d-optimal and breaks ties in order. (G, S_N) is sealed-bid and credible if and only if (G, S_N) is an almost-first-price auction.

Proof overview. Suppose (G, S_N) is an almost-first-price auction. (G, S_N) is sealed-bid by definition. If the highest-priority bidder has placed his maximum bid, then the auctioneer has no discretion; every safe deviation sells the object to that bidder at his bid. Otherwise, no safe deviation can sell at a price below the reserve, and every safe deviation involves charging a bidder his bid, so it is optimal to sell the object by the rules. Thus, (G, S_N) is credible.

Suppose (G, S_N) is credible and sealed-bid. If, after some action by bidder i , there are two prices that i might pay, the auctioneer can safely deviate to charge the higher price. Thus, for each action that i might play, there is a unique price that i might pay if he wins (his bid).

Suppose i has the highest priority, and plays the action assigned to θ_i^K . Since (G, S_N) is d-optimal, i must win for sure. Moreover, since (G, S_N) is BIC, this must be i 's highest possible bid, or lower types of i would deviate to that action.

Suppose that i has the highest priority, but does not play the action assigned to θ_i^K . For every bidder j who has bid above the reserve, if j wins and pays his bid, then that observation has an innocent explanation, and if j loses then that observation also has an innocent explanation. Therefore, if the mechanism does not award the object to the highest bidder (above the reserve), the auctioneer has a profitable safe deviation. Thus, (G, S_N) is an almost-first-price auction. (The full proof is in the Appendix.) \square

3.2 Credible and strategy-proof optimal auctions

We now characterize credible and ‘strategy-proof’ optimal auctions.

Definition 11. (G, S_N) is **strategy-proof** if, for all $i \in N$, for all S'_{-i}

$$S_i \in \operatorname{argmax}_{S'_i} \mathbb{E}_{\theta_N} [u_i(S'_i, S'_{-i}, \theta_N)] \quad (7)$$

(G, S_N) has **threshold pricing** if, for some reserve price $\rho \in \Theta_i$ and tie-breaking order \triangleright :

1. i wins the object if and only if for all $j \neq i$, $\theta_i \triangleright \max_{j \neq i} \theta_j$ and $\theta_i \triangleright \rho$.
2. If i wins the object, i pays $\min \{ \theta'_i \in \Theta_i \mid \theta'_i \triangleright \max_{j \neq i} \theta_j \text{ and } \theta'_i \triangleright \rho \}$.

(G, S_N) is **d-strategy-proof** if it is strategy-proof and has threshold pricing.

Clearly, d-strategy-proofness is a sufficient condition for strategy-proofness. It is a necessary condition in the continuous limit, in the sense that (for regular distributions) strategy-proofness and d-optimality imply threshold pricing by the Green-Laffont-Holmström theorem (Green and Laffont, 1977; Holmström, 1979).¹¹

Definition 12. (G, S_N) is an **ascending auction** (with reserve price ρ) if:

1. All bidders start as **active**.
2. The **high bidder** is the active bidder with the highest bid that is weakly above ρ (breaking ties according to \triangleright).
3. At each non-terminal history, some active bidder i (other than the high bidder) is called to play, and he chooses between actions that place a **bid** in Θ_i and actions that **quit**.

¹¹One adapts the definition of threshold pricing to the continuous case by replacing min with inf. Since we consider extensive game forms, threshold pricing is not a sufficient condition for strategy-proofness. For example, if (G, S_N) is a modified second price auction, such that agents bid their values under S_N , but agent 2 observes agent 1’s bid before placing his own bid, then (G, S_N) has threshold pricing but is not strategy-proof.

- (a) Each bid is no more than is necessary for i to become the high bidder.
- (b) If i quits, then he is no longer active.
- (c) At each information set, there is a unique action that places a bid, with one exception: If the reserve has not yet been met, and there is exactly one active bidder left, there may be multiple actions that place bids.

4. i 's strategy specifies:

- (a) If i 's type is strictly below a bid, he does not place that bid.
- (b) If i 's type is weakly above ρ and there is no high bidder, he places a bid.
- (c) If i 's type is above the current high bid (breaking ties with \triangleright), he places a bid.¹²

5. The auction ends if one of three conditions obtains:

- (a) If there are no active bidders. In that case, the object is not sold.
- (b) If only the high bidder is active. In that case, the object is sold to the high bidder at his last bid.
- (c) If the high bidder has bid θ^K , and no active bidder has higher tie-breaking priority. In that case, the object is sold to the high bidder at his last bid.

We pause to note a mild indeterminacy: When an active bidder is called to play, it could be that the available bid is not yet enough to become the high bidder. For instance, bidder i might choose whether to place a bid of 50 or quit, even though the current high bid is 100. In that case, types of i between 50 and 100 could place the bid or could quit. However, their actions are constrained by d-optimality, threshold pricing, and the fact that S_i must be measurable with respect to i 's information sets - it must be that bidder i never quits when he might still win.

The next theorem shows that, in the class of auctions we consider, the ascending auction is the unique credible, d-strategy-proof extensive form mechanism.

Theorem 2. *Assume F is regular and (G, S_N) is d-optimal and breaks ties in order. (G, S_N) is d-strategy-proof and credible if and only if (G, S_N) is an ascending auction.*

Proof overview. Suppose (G, S_N) is d-strategy-proof and credible. To prove that (G, S_N) is an ascending auction, we must show that for *any* extensive form that is not an ascending auction, there exists a profitable safe deviation for the auctioneer. The key is to see that ascending auctions have a close relationship to winner privacy (Milgrom and Segal, 2017) - at any history, if two types of agent i might both win the object, and some other agent might win the object instead, then both types of i must take the same action. If

¹²Notice that, since i 's strategy must be measurable with respect to i 's information sets, this implies that if i 's type is above the *least possible high bid associated with that information set*, he places a bid.

at some history this does not hold, then the auctioneer can exploit one type of agent i by deviating to charge him a higher price. In the case of a second-price auction, the auctioneer simply exaggerates the value of the second-highest bid. In general, however, the deviation must be more subtle in order to be safe - instead of just choosing a different outcome, the auctioneer may systematically misrepresent agents' actions midway through the extensive form. We construct an algorithm that produces a profitable safe deviation for any such extensive form.

Suppose (G, S_N) is an ascending auction. By inspection, it is d -strategy-proof. What remains is to show that it is credible. First, observe that (G, S_N) is not only d -optimal, but *fully* optimal - the local incentive constraints bind downward. Suppose that the auctioneer has a profitable safe deviation. For every agent i , S_i remains a best response to any safe deviation by the auctioneer. Thus, since the auctioneer has a profitable safe deviation, she can openly commit to that deviation without altering the agents' incentives - we can define a new protocol (G', S'_N) that is BIC and yields strictly more expected revenue than (G, S_N) . But (G, S_N) is optimal, so this is a contradiction. (The full proof is in the Appendix.) \square

By Theorem 1, restricting attention to revelation mechanisms forces a sharp choice between incentives for the auctioneer and strategy-proofness for the agents. Theorem 2 shows that allowing other extensive forms relaxes this trade-off.

3.3 An Auction Trilemma

Clearly, the almost-first-price auction is not strategy-proof, except in the degenerate case that only the highest type is above the optimal reserve. Thus, Theorems 1 and 2 yield the following corollary.

Corollary 1 (Auction Trilemma). *Assume F is regular and (G, S_N) breaks ties in order. Assume that there does not exist an optimal reserve ρ^* such that, for all $k < K$, $\theta_i^k \leq \rho^*$. No d -optimal (G, S_N) is sealed-bid, credible, and strategy-proof. However, there exist d -optimal (G, S_N) that are:*

1. *sealed-bid and strategy-proof (the second-price auction),*
2. *sealed-bid and credible (the first-price auction),*
3. *strategy-proof and credible (the ascending auction).*

Credibility, sealed bids, and strategy-proofness are all desirable properties. There is no optimal auction that has all three; and picking any two of three characterizes a classic auction format (Figure 2).

4 Asymmetric Distributions

So far we have assumed that bidders' values are independently and identically distributed. Suppose instead that type spaces and probability distributions could be asymmetric. That is, it may be that for agents i, j and some $k \in \mathbb{Z}$, $\theta_i^k \neq \theta_j^k$, and each agent has a probability mass function $p_i : \Theta_i \rightarrow [0, 1]$. Define f_i and F_i similarly, and

$$\eta_i(\theta_i) \equiv \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \quad (8)$$

Under asymmetry, first-price auctions remain credible but may no longer be optimal. In fact, there exist asymmetric type distributions such that no d-optimal (G, S_N) is sealed-bid and credible.

Example 1 (A d-optimal sealed-bid auction with asymmetric bidders is not credible). *Consider an auction with two bidders, where bidder 1's valuation has a uniform distribution with support $\Theta_1 = \{1, 2, \dots, 10\}$ and bidder 2's valuation also has a uniform distribution, but with support $\Theta_2 = \{11, 12, \dots, 20\}$. Thus, the virtual values are $\eta_1(\theta_1) = 2\theta_1 - 10$ for bidder 1 and $\eta_2(\theta_2) = 2\theta_2 - 20$ for bidder 2. By the same arguments as in the proof of Theorem 1, in a sealed-bid auction there must be a unique (real-valued) bid associated with each action.*

Now suppose bidder 1's type is $\theta_1 = 10$ and bidder 2's type is $\theta_2 = 11$. Any d-optimal auction must allocate the object to bidder 1 in this case, since $\eta_1(10) = 10$ and $\eta_2(11) = 2$. But note that bidder 1 can never pay more than 10 (to satisfy the IC constraint), while bidder 2's lowest type will never pay less than 11 (so that IR-0 binds). This, then, means that the auctioneer should allocate the object to the bidder with the lower bid. This is not credible because the auctioneer has a strictly profitable safe deviation: She can sell the object to bidder 2 (despite the fact that bidder 2 has a lower virtual value). Any $\theta_2 > 15$ would be an innocent explanation for bidder 1, and any $\theta_1 < 6$ would be an innocent explanation for bidder 2.

Example 1 demonstrates that, under asymmetry, the best sealed-bid credible mechanism may deliver strictly less revenue than the optimum.¹³

The next natural question is: When bidders are asymmetric, can any credible auction attain the optimum? It turns out that not only is there a credible optimal auction, there is even a *strategy-proof* credible optimal auction.

We modify the ascending auction of Definition 12, instead scoring bids according to their virtual value.

Definition 13. (G, S_N) is a **virtual ascending auction** if:

¹³This example can be modified so that both bidders' type distributions have the same support, but disjoint supports simplify the argument.

1. All bidders start as **active**, and the initial bids are $(b_i)_{i \in N} := (\theta_i^0)_{i \in N}$.
2. The **high bidder** is the active bidder with the highest **virtual value** $\eta_i(b_i)$ that is non-negative (breaking ties according to \triangleright).
3. At each non-terminal history, some active bidder i (other than the high bidder) is called to play, and he chooses either to place a **bid** in Θ_i or to **quit**.
 - (a) At each information set, there is a unique action that places a bid.
 - (b) Each bid is no more than is necessary for i to become the high bidder.
 - (c) If i quits, then he is no longer active.
4. i 's strategy specifies that he places a bid if and only if his type is weakly more than that bid.
5. The auction ends if one of three conditions obtains:
 - (a) If there are no active bidders. In that case, the object is not sold.
 - (b) If only the high bidder is active. In that case, the object is sold to the high bidder at his last bid.
 - (c) If the high bidder i has bid θ_i^K , and no active bidder can make a bid with a strictly higher virtual value. In that case, the object is sold to the high bidder at his last bid.

To illustrate, consider the type distributions of Example 1. We can run a virtual ascending auction in the following way: At the beginning, the virtual value of bidder 2 is at least $2 \times 11 - 20 = 2$ so he is the current high bidder. We first rule out all types of bidder 1 with negative virtual value by asking bidder 1 whether he is willing to pay at least 6. If he quits, bidder 2 wins and pays 11. If bidder 1 stays in, then we know his virtual value is at least as high as bidder 2's minimum virtual value, so he becomes the new high bidder. We then alternate between the bidders, asking each to increase his bid by 1. This implements the optimal allocation.

It is easy to verify that the virtual ascending auction is optimal and strategy-proof. To verify that it is credible as well, note that our proof strategy for the symmetric case applies *mutatis mutandis* to this case as well. The virtual ascending auction is optimal, and the agents' strategies are a best response to any safe deviation. Thus, if the auctioneer has a profitable safe deviation, she could pre-commit to that deviation, producing a new BIC protocol that yields strictly more expected revenue than the optimum, which is a contradiction.

In summary, asymmetry interacts with credibility in subtle ways. Clearly, when type distributions are asymmetric, standard first-price auctions and ascending auctions may

not be optimal. First-price auctions remain credible¹⁴, but Example 1 shows that there may be no way to modify them to restore optimality (while retaining credibility and sealed bids). Ascending auctions may not even be credible - the auctioneer may profitably deviate by running agents’ price clocks at different rates. However, modifying the scoring rule restores both credibility and optimality, while retaining strategy-proofness.

The foregoing arguments suggest another reason to look outside the class of sealed-bid mechanisms: General extensive forms allow the auctioneer to credibly commit to treat bidders asymmetrically.¹⁵ For *ex post* individually rational auctions with asymmetric distributions, the best credible auction can yield strictly more revenue than the best sealed-bid credible auction.

5 A ‘prior-free’ definition

The definition of credibility depends on the joint distribution of agent types (Definition 6). It may be useful to have a definition that is ‘prior-free’, for settings such as matching or maxmin mechanism design.

Definition 14. *Given (G, S_N) , $S_0 \in \mathcal{S}_0^*(S_0^G)$ is **profitable ex post** if, for all θ_N :*

$$u_0(S_0, S_N, \theta_N) \geq u_0(S_0^G, S_N, \theta_N) \tag{9}$$

with strict inequality for some θ_N .

*(G, S_N) is **weakly credible** if no safe deviation is profitable ex post.*

For comparison, (G, S_N) is credible if no safe deviation is profitable in expectation. Weak credibility allows one to dispense with strong assumptions about the auctioneer’s beliefs, in accordance with the Wilson Doctrine.¹⁶

What happens if we replace “credible” with “weakly credible” in the statement of Theorems 1 and 2? Since any credible protocol is weakly credible, this weakens one direction of implication but strengthens the other. Weak credibility still suffices to characterize both auctions.

Theorem 1*. *Assume F is regular and (G, S_N) is d -optimal and breaks ties in order. (G, S_N) is sealed-bid and weakly credible if and only if (G, S_N) is an almost-first-price auction.*

¹⁴First-price auctions are even ‘robustly’ credible, in the sense that they are credible regardless of the joint distribution of types.

¹⁵However, when one considers worst-case equilibria, standard first-price auctions have good revenue guarantees (Roughgarden, 2009; Feldman et al., 2016; Bergemann et al., 2017).

¹⁶Wilson (1987) writes, “Game Theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one player’s probability assessment about another’s preferences or information.”

Theorem 2*. Assume F is regular and (G, S_N) is d -optimal and breaks ties in order. (G, S_N) is d -strategy-proof and weakly credible if and only if (G, S_N) is an ascending auction.

Proof. Since credible protocols are weakly credible, the “if” direction is implied by Theorems 1 and 2.

In the proof of Theorem 1, we show that if (G, S_N) is sealed-bid but not an almost-first-price auction, then there exists a safe deviation that is profitable *ex post*. This proves the “only if” direction of Theorem 1*.

In the proof of Theorem 2, we show that if (G, S_N) is d -strategy-proof but not an ascending auction, then there exists a safe deviation that is profitable *ex post*. This proves the “only if” direction of Theorem 2*. \square

6 Discussion

To raise the old question once more: What accounts for the popularity of common real-world auction forms? “What determines which form will (or should) be used in any particular circumstance?” (Milgrom and Weber, 1982)

Our analysis indicates that the first-price auction and the ascending auction are not just historical accidents, but are game-theoretic solutions to a well-defined commitment problem. Furthermore, each auction form is the unique solution in its class.¹⁷

It is worth considering why real-world auctioneers might lack full commitment power. Vickrey (1961) suggests that the seller could delegate the task of running the auction to a third-party who has no stake in the outcome. However, auction houses such as Sotheby’s, Christie’s, and eBay charge commissions that are linear (or piecewise-linear) functions of the sale price.¹⁸ Running an auction takes effort, and many dimensions of effort are not contractible. Robust contracts reward the auctioneer linearly with revenue (Carroll, 2015), so it is difficult to employ a third-party who is both neutral and well-motivated.

When an auctioneer makes repeated sales, reputation could help enforce the full-commitment outcome. However, the force of reputation depends on the discount rate and the detection rate of deviations. Safe deviations are precisely those that a bidder could not detect immediately. Online advertising auctions are repeated frequently¹⁹, so it is plausible that bidders could examine the statistics to detect foul play. However, some economically important auctions are infrequent or not repeated at all - for instance,

¹⁷We have restricted attention to pruned protocols. One can produce other credible protocols by adding off-path histories or other inessential features.

¹⁸<http://www.sothebys.com/en/news-video/blogs/all-blogs/sotheby-s-at-large/2016/10/important-update-regarding-sothebys-buyers-premium.html>, <http://www.christies.com/buying-services/buying-guide/financial-information/>, <http://pages.ebay.com/help/sell/fees.html>, accessed 11/5/2017

¹⁹Edelman et al. (2007).

auctions for wireless spectrum or for the privatization of state-owned industries. Even established auction houses such as Christie’s and Sotheby’s have faced regulatory scrutiny, based in part on concerns that certain deviations are difficult for individual bidders to detect.

Not all auctioneers have full commitment power, just as not all firms are Stackelberg leaders. When the auctioneer lacks full commitment, it can be hazardous for bidders to reveal all their information at once. In a first-price auction, a bidder ‘reveals’ his value in return for a guarantee that his report completely determines the price he might pay.²⁰ In an ascending auction, a bidder reports whether his value is above b only when the auctioneer (correctly) asserts that bids below b are not enough to win. Credibility is a shared foundation for these seemingly disparate design features. How this principle extends to other environments is an open question.

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²⁰This property is generalized in a natural way by the ‘first-price’ menu auction (Bernheim and Whinston, 1986).

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A Definition of Extensive Game Forms with Consequences in X

An *extensive game form with consequences in X* is a tuple $(H, \prec, P, A, \mathcal{A}, (\mathcal{I}_i)_{i \in N}, g)$, where:

1. H is a set of histories, along with a binary relation \prec on H that represents precedence.
 - (a) \prec is a partial order, and (H, \prec) form an arborescence²¹.
 - (b) h_\emptyset denotes $h \in H : \neg \exists h' : h' \prec h$
 - (c) $Z \equiv \{h \in H : \neg \exists h' : h \prec h'\}$
 - (d) $\sigma(h)$ denotes the set of immediate successors of h .
2. P is a player function. $P : H \setminus Z \rightarrow N$.
3. A is a set of actions.
4. $\mathcal{A} : H \setminus h_\emptyset \rightarrow A$ labels each non-initial history with the last action taken to reach it.
 - (a) For all h , \mathcal{A} is one-to-one on $\sigma(h)$.
 - (b) $A(h)$ denotes the actions available at h .

$$A(h) \equiv \bigcup_{h' \in \sigma(h)} \mathcal{A}(h') \tag{10}$$

²¹That is, a directed rooted tree such that every edge points away from the root.

5. \mathcal{I}_i is a partition of $\{h : P(h) = i\}$ such that:

- (a) $A(h) = A(h')$ whenever h and h' are in the same cell of the partition.
- (b) For any $I_i \in \mathcal{I}_i$, we denote: $P(I_i) \equiv P(h)$ for any $h \in I_i$. $A(I_i) \equiv A(h)$ for any $h \in I_i$.
- (c) Each action is available at only one information set: If $a \in A(I_i)$, $a' \in A(I'_j)$, $I_i \neq I'_j$ then $a \neq a'$.

6. g is an outcome function. It associates each terminal history with an outcome.
 $g : Z \rightarrow X$

B Proofs omitted from the main text

B.1 Proposition 1

For each of the three clauses in Definition 2, we show that if (G, S_N) does not satisfy this clause, we can transform (G, S_N) to have strictly fewer histories, such that the transformed protocol is BIC and results in the same outcomes for each type profile. Since the set of histories in (G, S_N) is finite, it follows that there exists a BIC (G', S'_N) that cannot be reduced further, but results in the same outcomes as (G, S_N) .

Suppose there exists h such that there is no θ_N such that $h \preceq z^G(S_N, \theta_N)$. Since the game tree is finite, we can locate an earliest possible h ; that is, an h such that no predecessor satisfies this property. Consider h' that immediately precedes h , and the information set I'_i such that $h \in I'_i$. There is some action a' at I'_i that is not played by any type of i that reaches I'_i . We can delete all histories that follow i playing a' at I'_i (and define $(\prec', \mathcal{A}', P', (\mathcal{I}'_i)_{i \in N}, g')$ and S'_N so that they are as in G , but restricted to the new smaller set of histories H'). Since these histories were off the path of play, their deletion does not affect the incentives of agents in $N \setminus i$. Since i preferred his original S_i to any strategy that played a' at I'_i , his new strategy S'_i remains incentive-compatible. Thus, the transformed (G', S'_N) is BIC.

Suppose there exists $h \notin Z$ such that $|\sigma(h)| = 1$. We simply rewrite the transformed game (G', S'_N) that deletes h (and all the other histories in that same information set) and ‘automates’ i ’s singleton action at h . That is, for all $h' \in I_i$ for I_i such that $h \in I_i$, we remove h' from the set of histories, and define $(\prec', \mathcal{A}', P', (\mathcal{I}'_i)_{i \in N}, g')$ and S'_N so that they are as in G , but restricted to $H \setminus I_i$. (G', S'_N) is BIC.

Suppose there exists $h \notin Z$, such that for $i = P(h)$, there does not exist $\theta_i, \theta'_i, \theta_{-i}$ such that

1. $h \prec z^G(S_N, (\theta_i, \theta_{-i}))$
2. $h \prec z^G(S_N, (\theta'_i, \theta_{-i}))$

$$3. x^G(S_N, (\theta_i, \theta_{-i})) \neq x^G(S_N, (\theta'_i, \theta_{-i}))$$

If there does not even exist (θ_i, θ_{-i}) such that $h \prec z^G(S_N, (\theta_i, \theta_{-i}))$, then our argument about the first clause applies. Suppose there exists (θ_i, θ_{-i}) such that $h \prec z^G(S_N, (\theta_i, \theta_{-i}))$. Upon reaching h , we can henceforth ‘automate’ play as though i had type θ_i . That is, we can delete any history h' such that $h \preceq h'$ and $P(h') = i$, or $h \preceq h'$ and there does not exist θ''_{-i} such that $h' \prec z^G(S_N, (\theta_i, \theta''_{-i}))$. Given the new smaller set of histories H' , we again define $(\prec', \mathcal{A}', P', (\mathcal{I}'_i)_{i \in N}, g')$ and S'_N so that they are as in G , but restricted to H' .

By construction, for all θ'_i , if i is playing as though his type is θ'_i and we would have reached history h under (G, S_N) , then the outcome is the same under (G', S'_N) as when i is playing as though his type is θ_i under (G, S_N) (which by hypothesis is the same as when i is playing as though his type is θ'_i under (G, S_N)). Plainly, if we would not have reached history h under (G, S_N) , then the outcomes under (G, S_N) and (G', S'_N) are identical. Thus, (G', S'_N) is BIC.

This completes the proof of Proposition 1.

B.2 Proposition 2

To prove Proposition 2, we show that each of the three transformations we used in the proof of Proposition 1 also preserve credibility. That is, for each (G', S'_N) that is produced from (G, S_N) by one of the three transformations, if the auctioneer has a profitable safe deviation from $S_0^{G'}$, then she also has a profitable safe deviation from S_0^G .

Consider the first transformation (deleting all histories that follow action a at I_i , when a is never chosen on the path of play). Suppose the auctioneer had a profitable safe deviation S'_0 from $S_0^{G'}$. The auctioneer could make that same deviation, but additionally offer the response $\lambda(a)$ whenever sending the message $\lambda(I_i)$. By hypothesis, agent i never selects $\lambda(a)$ as a reply, so for any θ_N and any j , j 's resulting observation has an innocent explanation. Thus, the auctioneer also has a profitable safe deviation from S_0^G .

Consider the second transformation (deleting all histories in some information set with a singleton action set). Suppose the auctioneer had a profitable safe deviation S'_0 from $S_0^{G'}$. The auctioneer could make that same deviation from S_0^G , except that for the deleted information set I_i , the auctioneer delays sending $\lambda(I_i, A(I_i))$ until the last possible moment. That is consider S_0 that is the same as S'_0 , except that, if the auctioneer has not yet sent $\lambda(I_i, A(I_i))$ to agent i , then:

1. If S'_0 specifies that the auctioneer sends $\lambda(I'_i, A(I'_i))$ for $I'_i \succ I_i$, then S_0 specifies that she first sends $\lambda(I_i, A(I_i))$ and then (immediately thereafter) sends $\lambda(I'_i, A(I'_i))$
2. If S'_0 specifies that the auctioneer chooses an outcome such that the resulting observation for i does not have an innocent explanation under S_0^G , then S_0 specifies that she first sends $\lambda(I_i, A(I_i))$ before choosing that outcome.

S_0 is a profitable safe deviation from S_0^G .

Consider the third transformation (deleting histories where i is called to play, following some history h such that, for any two types of i that reach h , both types of i result in the same outcome). Suppose S_0' was a profitable safe deviation from $S_0^{G'}$.

If the observation for i that results from S_0' does not have an innocent explanation under S_0^G , it must be that (given on all the communication i has seen so far), the outcome S_0' is about to select can only occur under G at terminal histories that follow h . But by hypothesis, for any θ_i and θ'_i that are consistent with reaching h , and any θ_{-i} consistent with reaching h , the resulting outcome is the same. Thus, let S_0 be exactly as in S_0' , except that if S_0' specifies that the auctioneer chooses an outcome such that the resulting observation for i does not have an innocent explanation under S_0^G , then S_0 specifies that the auctioneer communicates with i as though play started from h and the opponent type profiles were θ_{-i} , for some θ_{-i} consistent with reaching h .

Formally, if S_0' would choose an outcome such that i 's observation has no innocent explanation, then fix some θ_N such that $h \prec z^G(S_N, \theta_N)$. Initialize $\hat{h} := h$.

1. If $\hat{h} \in Z$, then terminate and choose $x = g(\hat{h})$.
2. Else if $P(\hat{h}) \neq i$, then for $I_{P(\hat{h})}$ such that $\hat{h} \in I_{P(\hat{h})}$:
 - (a) $\hat{h} := h' \mid h' \in \sigma(\hat{h})$ and $S_{P(\hat{h})}(I_{P(\hat{h})}, \theta_{P(\hat{h})}) = \mathcal{A}(h')$.
 - (b) Go to step 1.
3. Else:
 - (a) Send $(m, R) = \lambda(I_i, A(I_i))$ for I_i such that $\hat{h} \in I_i$.
 - (b) Upon receiving $r \in R$, choose $\hat{h} := h' \mid \mathcal{A}(h') = \lambda^{-1}(r)$ and $h' \in \sigma(\hat{h})$.
 - (c) Go to step 1.

Since, under S_i , i 's play in this final stage makes no difference to the outcome, delaying communication with i until the outcome is about to be selected results in a safe deviation. This completes the proof of Proposition 2.

B.3 Proposition 4

Let $y_i^{G, S_N}(\theta_i, \theta_{-i})$ denote the probability i gets the object when his type is θ_i and his opponent types are θ_{-i} .

Thus,

$$\begin{aligned}
\check{u}_i^{G, S_N}(k, k) - \check{u}_i^{G, S_N}(k-1, k-1) \\
&= \check{u}_i^{G, S_N}(k, k-1) - \check{u}_i^{G, S_N}(k-1, k-1) \\
&= \epsilon \mathbf{E}_{\theta_{-i}}[y_i^{G, S_N}(\theta^{k-1}, \theta_{-i})] \quad (11)
\end{aligned}$$

$$\begin{aligned}\tilde{u}_i^{G,S_N}(k, k) &= \sum_{j=1}^k \epsilon \mathbf{E}_{\theta_{-i}}[y_i^{G,S_N}(\theta^{j-1}, \theta_{-i})] \\ &= \mathbf{E}_{\theta_{-i}}\left[\sum_{j=1}^k \epsilon y_i^{G,S_N}(\theta^{j-1}, \theta_{-i})\right] \quad (12)\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{\theta_i}(\mathbf{E}_{\theta_{-i}}(\text{revenue from } i)) \\ = \mathbf{E}_{\theta_{-i}}(\mathbf{E}_{\theta_i}(\text{welfare from } i)) - \mathbf{E}_{\theta_i}(\mathbf{E}_{\theta_{-i}}(\text{utility for } i)) \quad (13)\end{aligned}$$

$$\mathbf{E}_{\theta_{-i}}(\mathbf{E}_{\theta_i}(\text{welfare from } i)) = \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K \epsilon f(\theta^k) \theta^k y_i^{G,S_N}(\theta^k, \theta_{-i})\right] \quad (14)$$

$$\begin{aligned}\mathbf{E}_{\theta_i}(\mathbf{E}_{\theta_{-i}}(\text{utility for } i)) &= \mathbf{E}_{\theta_{-i}}\left[\sum_{k=1}^K \epsilon f(\theta^k) \sum_{j=1}^k \epsilon y_i^{G,S_N}(\theta^{j-1}, \theta_{-i})\right] \\ &= \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K (1 - F(\theta^k)) \epsilon y_i^{G,S_N}(\theta^k, \theta_{-i})\right] \\ &= \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K \epsilon f(\theta^k) \frac{1 - F(\theta^k)}{f(\theta^k)} y_i^{G,S_N}(\theta^k, \theta_{-i})\right] \quad (15)\end{aligned}$$

Combining Equations 14 and 15 yields:

$$\mathbf{E}_{\theta_i}(\mathbf{E}_{\theta_{-i}}(\text{revenue from } i)) = \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K \epsilon f(\theta^k) y_i^{G,S_N}(\theta^k, \theta_{-i}) \underbrace{\left[\theta^k - \frac{1 - F(\theta^k)}{f(\theta^k)}\right]}_{\text{virtual value}}\right] \quad (16)$$

Summing across agents yields Proposition 4.

B.4 Proposition 5

Since (G, S_N) is BIC, every type of i prefers playing according to S_i , instead of playing as though his type is ϵ higher. That is,

$$\forall i : \forall k > 1 : \tilde{u}_i^{G,S_N}(k-1, k-1) \geq \tilde{u}_i^{G,S_N}(k-1, k) \quad (17)$$

Since IR-0 binds,

$$\begin{aligned}
\check{u}_i^{G,S_N}(k,k) &= \sum_{l=1}^k \check{u}_i^{G,S_N}(l,l) - \check{u}_i^{G,S_N}(l-1,l-1) \\
&\leq \sum_{l=1}^k \check{u}_i^{G,S_N}(l,l) - \check{u}_i^{G,S_N}(l-1,l) = \sum_{l=1}^k \epsilon \mathbf{E}_{\theta_{-i}}[y_i^{G,S_N}(\theta^l, \theta_{-i})]
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathbf{E}_{\theta_i}(\mathbf{E}_{\theta_{-i}}(\text{utility for } i)) &\leq \mathbf{E}_{\theta_{-i}}\left[\sum_{k=1}^K \epsilon f(\theta^k) \sum_{j=1}^k \epsilon y_i^{G,S_N}(\theta^j, \theta_{-i})\right] \\
&= \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K (1 - F(\theta^k)) \epsilon y_i^{G,S_N}(\theta^k, \theta_{-i})\right] + \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K \epsilon f(\theta^k) (y_i^{G,S_N}(\theta^k, \theta_{-i}) - y_i^{G,S_N}(\theta^0, \theta_{-i}))\right] \\
&= \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K \epsilon f(\theta^k) \frac{1 - F(\theta^k)}{f(\theta^k)} y_i^{G,S_N}(\theta^k, \theta_{-i})\right] + \mathbf{E}_{\theta_{-i}}\left[\sum_{k=0}^K \epsilon f(\theta^k) (y_i^{G,S_N}(\theta^k, \theta_{-i}) - y_i^{G,S_N}(\theta^0, \theta_{-i}))\right]
\end{aligned} \tag{19}$$

Using the same steps as in Proposition 4 yields

$$\begin{aligned}
\pi(G, S_N) &\geq \\
&\mathbf{E}_{\theta_N}\left[\sum_{i \in N} y_i^{G,S_N}(\theta_N) \eta(\theta_i)\right] - \underbrace{\epsilon \mathbf{E}_{\theta_N}\left[\sum_{i \in N} y_i^{G,S_N}(\theta_N)\right]}_{\leq 1} - \underbrace{\sum_{i \in N} y_i^{G,S_N}(\theta_i^0, \theta_{-i})}_{\geq 0} \tag{20}
\end{aligned}$$

B.5 Theorem 1

Suppose (G, S_N) is an almost-first-price auction. (G, S_N) is sealed-bid by definition, so we need only establish that it is credible. Take S_0^G that runs G and the set of safe deviations of the auctioneer $\mathcal{S}_0^*(S_0^G)$. Every i has exactly one information set I_i and is always called to play, so the auctioneer is constrained to send $\lambda(I_i, A(I_i))$ before announcing an outcome. (Any order that she communicates with agents is optimal.) Once the auctioneer has messaged every agent, she faces some bid profile $(b_i(a_i))_{i \in N}$, and since the auctioneer is limited to the set of safe deviations, each i can only be charged $b_i(a_i)$, and only if $b_i(a_i) \geq \rho$. If i wins all ties and has type θ_i^K (submitting his highest possible bid $\max_{a'_i} b_i(a'_i)$), then i wins for certain, so the auctioneer has no room to deviate. Otherwise, if $\max b_i(a_i) \geq \rho$, then S_0 is optimal if and only if it sets $y \in \arg\max_i b_i(a_i)$. Thus, $S_0^G \in \arg\max_{S_0^G \in \mathcal{S}_0^*(S_0^G)} \mathbb{E}_{\theta_N}[u_0(S_0^G, S_N, \theta_N)]$ and (G, S_N) is credible.

Suppose (G, S_N) is sealed-bid and credible. We want to show that (G, S_N) is an almost-first-price auction. Take S_0^G that runs G and safe deviations $\mathcal{S}_0^*(S_0^G)$.

Suppose there exists i and $a_i \in A(I_i)$ such that after playing a_i , i could win the object at two different prices $b_i < b'_i$. Then for some action profile (a_i, a_{-i}) , S_0^G chooses outcome $(y, t) = (i, b_i)$. Consider $S'_0 \in \mathcal{S}_0^*(S_0^G)$ that is identical to S_0^G , except that when the action profile is (a_i, a_{-i}) , S'_0 sets $(y, t) = (i, b'_i)$, yielding higher revenue. Since (G, S_N) is pruned, S'_0 is a safe deviation and (G, S_N) is not credible.

Thus, for all i there exists $b_i : A(I_i) \rightarrow \mathbb{R}$ such that if i plays a_i and wins the object, then i pays $b_i(a_i)$. (We normalize this to $b_i(a_i) = -1$ if i never wins the object after playing a_i , and set ρ to be weakly more than 0 and weakly less than the least non-negative bid.) Suppose given bid profile $(b_j(a_j))_{j \in N}$, i wins the object even though $i \notin \operatorname{argmax}_j b_j(a_j)$. If for all $j \neq i$, $i \triangleright j$, and i 's type is θ_i^K , then i must win the object, since (G, S_N) is d-optimal and break ties in order. In that case, since (G, S_N) is pruned and BIC, it follows that $b_i(a_i) = \max_{a'_i} b_i(a'_i)$.

If $\theta_i \neq \theta_i^K$ or i does not win all ties, we exhibit a profitable deviation: S_0^G awards i the object when the bid profile is $(b_j(a_j))_{j \in N}$. Take $l \in N \setminus i$ such that $b_i(a_i) < b_l(a_l)$. Consider S'_0 that is identical to S_0 , except that it sets $(y, t) = (l, b_l(a_l))$ when the bid profile is $(b_j(a_j))_{j \in N}$. Since there exists θ'_{-i} such that $y_i^{G, S_N}(\theta_i, \theta'_{-i}) = 0$ and $b_l(a_l) > 0$, $S'_0 \in \mathcal{S}_0^*(S_0^G)$. Thus $S_0 \notin \operatorname{argmax}_{S'_0 \in \cap_{i \in N} \hat{\mathcal{S}}_0^i} \mathbb{E}_{\theta_N} [u_0(S'_0, S_N, \theta_N)]$ and (G, S_N) is not credible.

Thus, if (G, S_N) is sealed-bid and credible, then (G, S_N) is an almost-first-price auction.

B.6 Theorem 2

Given (G, S_N) , let Θ_i^h denote the types of i that are consistent with i 's actions up to history h , that is:

$$\Theta_i^h = \{\theta_i | \forall h', h'' \preceq h : [h' \in I_i, h'' \in \sigma(h')] \rightarrow [S_i(I_i, \theta_i) = \mathcal{A}(h'')]\} \quad (21)$$

Proposition 6. *If $h \prec h'$ then $\Theta_i^h \supseteq \Theta_i^{h'}$. If $h \in I_i$ and $h' \in I_i$, then $\Theta_i^h = \Theta_i^{h'}$.*

The first is clear by inspection. The second follows because the definition of Θ_i^h invokes only i 's past information sets and actions, and G has perfect recall. Thus, we define $\Theta_i^{I_i} = \Theta_i^h | h \in I_i$.

Define also:

$$\underline{\theta}_i^h = \min_{\theta_i \in \Theta_i^h} \quad (22)$$

$$\bar{\theta}_i^h = \max_{\theta_i \in \Theta_i^h} \quad (23)$$

B.6.1 credible, d-strategy-proof \rightarrow ascending

Since F is regular, and (G, S_N) is d-optimal and d-strategy-proof, the reserve price must be a d-optimal reserve price ρ^* .

We define the types of i that will win when facing types θ_{-i} and reserve ρ^* , that is

$$\Theta_i^{\text{win}}(\theta_{-i}, \rho^*) = \{\theta_i | \theta_i \triangleright \rho^* \text{ and } \theta_i \triangleright \max_{j \neq i} \theta_j\} \quad (24)$$

Let $\tau_i(\theta_{-i}, \rho)$ denote the price that i pays if he wins when the opponent types are θ_{-i} and the reserve price is ρ , where we set this equal to the highest possible type if i is unable to win. That is,

$$\tau_i(\theta_{-i}, \rho) = \begin{cases} \min \triangleright \Theta_i^{\text{win}}(\theta_{-i}, \rho) & \text{if } \Theta_i^{\text{win}}(\theta_{-i}, \rho) \neq \emptyset \\ \theta_i^K & \text{otherwise} \end{cases} \quad (25)$$

Lemma 1 (Mutual secrecy). *Assume (G, S_N) is d-strategy-proof with reserve price ρ . For any distinct $i, j \in N$, any I_j , $h \in I_j$, $\theta_{-i} \in \Theta_{-i}^h$. If $\Theta_i^h \cap \Theta_i^{\text{win}}(\theta_{-i}, \rho) \neq \emptyset$ and $\bar{\theta}_j^h \triangleright \tau_i(\theta_{-i}, \rho)$, then for all $\theta_i \in \Theta_i^{\text{win}}(\theta_{-i}, \rho)$, there exists $h' \in I_j$ such that $(\theta_i, \theta_{-i}) \in \Theta_N^{h'}$.*

Proof. Suppose the antecedent is true but the consequent is not. Pick $\theta_i \in \Theta_i^h \cap \Theta_i^{\text{win}}(\theta_{-i}, \rho)$. Pick $\theta'_i \in \Theta_i^{\text{win}}(\theta_{-i}, \rho)$ such that there does not exist $h' \in I_j$ such that $(\theta'_i, \theta_{-i}) \in \Theta_N^{h'}$. (Trivially, $\theta_i \neq \theta'_i$.) We will define S'_{-i} such that i has a profitable deviation, so (G, S_N) is not d-strategy-proof.

Suppose for now that $\theta_i \triangleleft \theta'_i$. For $l \in N \setminus \{i, j\}$, S_l specifies that l plays as though his type is θ_l . S_j specifies that j plays as though his type is $\bar{\theta}_j^h$, unless he encounters I_j . Upon encountering I_j , j plays thereafter as though his type is θ_j . i now has a profitable deviation. When i 's type is θ'_i , playing S_i means that j never encounters I_j , so i has payoff $\max\{0, \theta'_i - \tau_i(\bar{\theta}_j^h, \theta_{N \setminus \{i, j\}}), \rho\}$. On the other hand, if i deviates to play as though his type is θ_i , then since $(\bar{\theta}_j^h, \theta_{N \setminus \{i, j\}}) \in \Theta_{-i}^h$, j encounters I_j with certainty. Thus i 's payoff is $\theta'_i - \tau_i(\theta_{-i}, \rho) > \max\{0, \theta'_i - \tau_i(\bar{\theta}_j^h, \theta_{N \setminus \{i, j\}}), \rho\}$, where the strict inequality follows since $\theta'_i \triangleright \tau_i(\theta_{-i}, \rho)$ and $\bar{\theta}_j^h \triangleright \tau_i(\theta_{-i}, \rho)$.

Suppose instead that $\theta_i \triangleright \theta'_i$. For $l \in N \setminus \{i, j\}$, S_l specifies that l plays as though his type is θ_l . S_j specifies that j plays as though his type is θ_j , unless he encounters I_j . Upon encountering I_j , j plays thereafter as though his type is $\bar{\theta}_j^h$. When i 's type is θ_i , playing S_i means that j encounters I_j with certainty, so i has payoff $\max\{0, \theta_i - \tau_i(\bar{\theta}_j^h, \theta_{N \setminus \{i, j\}}), \rho\}$. On the other hand, if i deviates to play as though his type is θ'_i , then j never encounters I_j . Thus i 's payoff is $\theta_i - \tau_i(\theta_{-i}, \rho) > \max\{0, \theta_i - \tau_i(\bar{\theta}_j^h, \theta_{N \setminus \{i, j\}}), \rho\}$, where the strict inequality follows since $\theta_i \triangleright \tau_i(\theta_{-i}, \rho)$ and $\bar{\theta}_j^h \triangleright \tau_i(\theta_{-i}, \rho)$. \square

Lemma 2 (Potential winners pool). *Assume (G, S_N) is d-optimal, d-strategy-proof and credible. For all I_i , $h \in I_i$, $\theta_i \in \Theta_i^h$, if $\theta_i \triangleright \tau_i(\underline{\theta}_{-i}^h, \rho^*)$ and there exists $j \neq i$ such that $\bar{\theta}_j^h \triangleright \tau_i(\underline{\theta}_{-i}^h, \rho^*)$, then $S_i(I_i, \theta_i) = S_i(I_i, \tau_i(\underline{\theta}_{-i}^h, \rho^*))$.*

Proof. Suppose not. Take some earliest h and I_i such that Lemma 2 does not hold; i.e. h such that for all $h' \prec h$, Lemma 2 holds at h' . We will exhibit a profitable safe deviation that the auctioneer can undertake, upon encountering h while running G .

Define $\theta_i^* \equiv \min^{\triangleright} \{\theta_i \mid \theta_i \triangleright \tau_i(\underline{\theta}_{-i}^h, \rho^*) \text{ and } S_i(I_i, \theta_i) \neq S_i(I_i, \tau_i(\underline{\theta}_{-i}^h, \rho^*))\}$. Note that since Lemma 2 holds for every $h' \prec h$, $\theta_i^* \in \Theta_i^h$. Define h^* to be the immediate successor of h such that $\theta_i^* \in \Theta_i^{h^*}$.

Consider the agents who might still win the object conditional on reaching h , that is, $N^* \equiv \{j \in N \mid \Theta_j^{\text{win}}(\underline{\theta}_{-j}^h, \rho^*) \cap \Theta_j^h \neq \emptyset\}$. For each agent j , we pick a ‘nemesis’; this is the highest-priority agent in $N^* \setminus j$. That is, $\psi(j) \equiv \max^{\triangleright} N^* \setminus j$.²²

Since Lemma 2 holds for every $h' \prec h$, for all $j \in N^* \setminus \{i\}$, $\Theta_j^{\text{win}}(\underline{\theta}_{-j}^h, \rho^*) \subseteq \Theta_j^h$.

Consider, for any type θ , the least upper bound type of j , that is $\min\{\theta_j \mid \theta_j \succeq \theta\}$. We can then define the set (equivalence class) of types such that have the same least upper bound. Formally:

$$\text{CLASS}_j(\theta) \equiv \begin{cases} \{\theta' \mid \min^{\triangleright} \{\theta_j \mid \theta_j \succeq \theta'\} = \min^{\triangleright} \{\theta_j \mid \theta_j \succeq \theta\}\} & \text{if } \{\theta_j \mid \theta_j \succeq \theta\} \neq \emptyset \\ \{\theta' \mid \theta' \triangleright \theta_j^K\} & \text{otherwise} \end{cases}. \quad (26)$$

Given S_0^G (with corresponding λ), we now exhibit a (partial) behavioral strategy that deviates from S_0^G upon encountering h^* and is strictly profitable. We describe this algorithmically. (The description is lengthy, because it must produce a safe deviation for any extensive game form in a large class.)

The algorithm is divided into four parts; **First-Query**, **Safety**, **Query**, and **Conclude**. Essentially, the auctioneer first pauses communication with $N \setminus i$ and checks whether i ’s type is at least θ_i^* (**First-Query**). If i ’s type is revealed to be less than θ_i^* , that implies that i would not win under S_0^G . In that case, the auctioneer communicates with $N \setminus i$ in a way that ensures her payoff is at least as high as under S_0^G (**Safety**). If i ’s type is revealed to be at least θ_i^* , the auctioneer then treats θ_i^* as her ‘best offer’, and communicates with agents in turn seeing if any agent can beat the best offer, updating the best offer whenever she finds a better offer (**Query**). She continues this until the agent with the best offer is the only agent left, whereupon she sells to that agent at a price equal to the best offer (**Conclude**).

This deviation always results in revenue at least as high as under S_0^G , and with positive probability results in strictly more revenue - in particular, when the type profile is $(\theta_i^*, \underline{\theta}_{-i}^{h^*})$. The key is to write this down precisely, and to show that the deviation can be carried out safely.

We use the notation $:=$ for the assignment operator, and we use $:\in$ to assign an (arbitrary) element from the set on the right-hand side.

²²This choice of nemesis deals with boundary cases; it ensures that if there is *any* offer from an agent in N^* that j cannot beat because of the tie-breaking rule, we can pick a nemesis type that j cannot beat.

This algorithm keeps track of:

1. A best offer, initialized $\beta := \theta_i^*$.
2. A set of ‘active’ agents, initialized $\hat{N} := N$.
3. A current agent called to play, $\hat{j} := i$.
4. A simulated history, $\hat{h} := h^*$.
5. The last information set each agent saw, if it exists, $(\hat{I}_j)_{j \in N}$.

$$\hat{I}_j := I_j \prec h^* \mid \forall I'_j \prec h^* : I'_j \preceq I_j \quad (27)$$

6. The last action each agent took, $(\hat{a}_j)_{j \in N}$, initialized to be the action taken at \hat{I}_j .

We start at Step 1 of **First-Query**.

First-Query

1. If $\underline{\theta}_i^{\hat{h}} \geq \beta$, then:

- (a) $\beta := \underline{\theta}_i^{\hat{h}}$
- (b) Assign (\hat{I}_i, \hat{a}_i) to be the latest information set and action that i encountered.
- (c) $\hat{j} := \hat{N} \setminus i$.
- (d) If $\hat{a}_{\hat{j}} \neq \emptyset$, then for $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$,

$$\hat{h} := h \mid h \in \sigma(\hat{I}_{\hat{j}}) \text{ and } \mathcal{A}(h) = \hat{a}_{\hat{j}} \text{ and } (\theta_{\psi(\hat{j})}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}) \in \Theta_{-\hat{j}}^h \quad (28)$$

- (e) Else $\hat{h} := h_0$.

- (f) Go to **Query**.

2. Else if $\hat{h} \in Z$, then $\beta := \bar{\theta}_i^{\hat{h}}$, $\hat{h} := h^*$, and go to **Safety**.²³

3. Else if $P(\hat{h}) = i$

- (a) Send $\lambda(I_i, A(I_i))$ for $I_i \mid \hat{h} \in I_i$ to i .
- (b) Upon receiving r , set $\hat{h} := h' \mid h' \in \sigma(\hat{h})$ and $\mathcal{A}(h') = \lambda^{-1}(r)$.
- (c) Go to step 1.

4. Else:

²³Here we use β to keep track of the highest type consistent with i 's actions, which we have learned is not enough to win.

(a) For $\theta_{\psi(i)} \in \text{CLASS}_i(\beta)$

$$\hat{h} := h \mid h \in \sigma(\hat{h}) \text{ and } (\theta_{\psi(i)}, \underline{\theta}_{N \setminus \{i, \psi(i)\}}^{h*}) \in \Theta_{-i}^h \quad (29)$$

(b) Go to step 1.

Query

1. If $|\hat{N}| = 1$, go to **Conclude**.

2. If $\underline{\theta}_{\hat{j}}^{\hat{h}} \geq \beta$, then:

(a) $\beta := \underline{\theta}_{\hat{j}}^{\hat{h}}$

(b) Assign $(\hat{I}_{\hat{j}}, \hat{a}_{\hat{j}})$ to be the latest information set and action that \hat{j} encountered.

(c) $\hat{j} := \hat{N} \setminus \hat{j}$.

(d) If $\hat{a}_{\hat{j}} \neq \emptyset$, then for $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$,

$$\hat{h} := h \mid h \in \sigma(\hat{I}_{\hat{j}}) \text{ and } \mathcal{A}(h) = \hat{a}_{\hat{j}} \text{ and } (\theta_{\psi(\hat{j})}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h*}) \in \Theta_{-\hat{j}}^h \quad (30)$$

(e) Else $\hat{h} := h_0$.

(f) Go to step 1.

3. Else if $\hat{h} \in Z$, then:

(a) $\hat{N} := \hat{N} \setminus \hat{j}$.

(b) $\hat{j} := \hat{N}$.

(c) If $\hat{a}_{\hat{j}} \neq \emptyset$, then for $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$,

$$\hat{h} := h \mid h \in \sigma(\hat{I}_{\hat{j}}) \text{ and } \mathcal{A}(h) = \hat{a}_{\hat{j}} \text{ and } (\theta_{\psi(\hat{j})}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h*}) \in \Theta_{-\hat{j}}^h \quad (31)$$

(d) Else $\hat{h} := h_0$.

(e) Go to step 1.

4. Else if $P(\hat{h}) = \hat{j}$, then:

(a) Send $\lambda(I_{\hat{j}}, A(I_{\hat{j}}))$ for $I_{\hat{j}} \mid \hat{h} \in I_{\hat{j}}$ to \hat{j} .

(b) Upon receiving r , set $\hat{h} := h' \mid h' \in \sigma(\hat{h})$ and $\mathcal{A}(h') = \lambda^{-1}(r)$.

(c) Go to step 1.

5. Else:

(a) For $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$

$$\hat{h} := h \mid h \in \sigma(\hat{h}) \text{ and } (\theta_{\psi(\hat{j})}, \theta_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h*}) \in \Theta_{-\hat{j}}^h \quad (32)$$

(b) Go to step 1.

Safety

1. If $\hat{h} \in Z$, then terminate and choose $x = g(\hat{h})$.

2. Else if $P(\hat{h}) = i$:

(a) $\hat{h} := h \mid h \in \sigma(\hat{h})$ and $\beta \in \Theta_i^h$.

(b) Go to step 1.

3. Else:

(a) Choose agent $P(\hat{h})$ and send $(m, R) = \lambda(I_j, A(I_j))$ for I_j such that $\hat{h} \in I_j$.

(b) Upon receiving $r \in R$, choose $\hat{h} := h' \mid \mathcal{A}(h') = \lambda^{-1}(r)$ and $h' \in \sigma(\hat{h})$.

(c) Go to step 1.

Conclude

1. If $\hat{h} \in Z$, then terminate and choose $x = g(\hat{h})$.

2. Else if $P(\hat{h}) = \hat{j}$, then:

(a) Send $\lambda(I_{\hat{j}}, A(I_{\hat{j}}))$ for $I_{\hat{j}} \mid \hat{h} \in I_{\hat{j}}$ to \hat{j} .

(b) Upon receiving r , set $\hat{h} := h' \mid h' \in \sigma(\hat{h})$ and $\mathcal{A}(h') = \lambda^{-1}(r)$.

(c) Go to step 1.

3. Else:

(a) For $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$

$$\hat{h} := h \mid h \in \sigma(\hat{h}) \text{ and } (\theta_{\psi(\hat{j})}, \theta_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h*}) \in \Theta_{-\hat{j}}^h \quad (33)$$

(b) Go to step 1.

The algorithm terminates in two possible ways. Either it goes from **First-Query** to **Safety**, or it goes from **First-Query** to **Query** to **Conclude**.

If the algorithm goes from **First-Query** to **Safety**, then agent i has (by construction) observed a communication sequence consistent with some terminal history z at which he does not win. All the other agents observe a communication sequence consistent with i

playing as though his type is $\bar{\theta}_i^z$. Since $h^* \prec z$, Proposition 6 implies that $\bar{\theta}_i^z \in \Theta_i^{h^*}$, so every agent's observation has an innocent explanation. Moreover, i 's true type is no more than $\bar{\theta}_i^z$, which is strictly less than the least type i would need to win at the non-pooling history where we started. Thus, if the algorithm goes from **First-Query** to **Safety**, revenue is at least as high as it would have been under S_0^G .

If the algorithm goes from **First-Query** to **Query** to **Conclude**, notice that whenever we initiate communication with an agent, if he has seen any communication before, then we pick a simulated history that is among the immediate successors of the last information set he saw, and consistent with the last action that he took (Step 1.d of **First-Query** and steps 2.d and 3.c of **Query**). Moreover, whenever an agent is removed from the set \hat{N} during **Query**, he has already seen a communication sequence consistent with a terminal history at which he does not win (Step 3). Finally, **Conclude** ensures that the agent who does win the object sees a communication sequence consistent with that happening.

It remains to show that we can in fact pick simulated histories in this way; *i.e.* that Step 1.d of **First-Query** and steps 2.d and 3.c of **Query** are well-defined. Consider two cases; the first time the deviating algorithm communicates with an agent, and all subsequent times.

The first time the deviating algorithm communicates with an agent \hat{j} ,²⁴ we need to pick a history consistent with $(\theta_{\psi(\hat{j})}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*})$ for $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$. If \hat{j} has not seen any information sets yet, then h_0 will do. Suppose \hat{j} was called to play at some history h' before we started deviating at h . By hypothesis, Lemma 2 holds at h' . Moreover, for all $l \in N^* \setminus \{\hat{j}\}$, $\Theta_l^{\text{win}}(\underline{\theta}_{-l}^h, \rho^*) \subseteq \Theta_l^h \subseteq \Theta_l^{h'}$, where the second set inclusion is by Proposition 6. In particular, since \hat{j} 's nemesis $\psi(\hat{j})$ was picked from $N^* \setminus \{\hat{j}\}$, and $\beta \geq \theta_i^* \geq \underline{\theta}_{\psi(\hat{j})}^{h'}$, this implies that for $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$, $(\theta_{\psi(\hat{j})}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}) \in \Theta_{-\hat{j}}^{h'}$. Thus, Step 1.d of **First-Query** and steps 2.d and 3.c of **Query** are well-defined for this case.

Consider any subsequent time that the deviating algorithm initiates communication with agent \hat{j} . By construction of **First-Query** and **Query**, this implies that the agent has previously been queried and beaten the (then) best offer β^{old} . We have to show that we can pick a history that immediately succeeds $\hat{I}_{\hat{j}}$ that is consistent with $(\theta_{\psi(\hat{j})}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*})$ for $\theta_{\psi(\hat{j})} \in \text{CLASS}_{\hat{j}}(\beta)$, for $\beta \geq \beta^{\text{old}}$.

When the auctioneer last communicated with \hat{j} , she reached a simulated history $h^{\text{old}} \in \hat{I}_{\hat{j}}$, where $(\theta_{\psi(\hat{j})}^{\text{old}}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}) \in \Theta_{-\hat{j}}^{h^{\text{old}}}$ for $\theta_{\psi(\hat{j})}^{\text{old}} \in \text{CLASS}_{\hat{j}}(\beta^{\text{old}})$. This is the simulated history at which \hat{j} was last called to play, and then took an action that revealed that his type beat $\theta_{\psi(\hat{j})}^{\text{old}}$.

Since at h^{old} , \hat{j} has *not yet* beaten $\theta_{\psi(\hat{j})}^{\text{old}}$, it follows that

$$\Theta_{\psi(\hat{j})}^{h^{\text{old}}} \cap \Theta_{\psi(\hat{j})}^{\text{win}}((\theta_{-\hat{j}}^{h^{\text{old}}}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}), \rho^*) \neq \emptyset \quad (34)$$

²⁴That is, the first time that we perform Step 1.d of **First-Query** or steps 2.d or 3.c of **Query** for that agent.

Since \hat{j} 's action at $\hat{I}_{\hat{j}}$ revealed that his type beats $\theta_{\psi(\hat{j})}^{\text{old}}$, it follows that

$$\bar{\theta}_{\hat{j}}^{h^{\text{old}}} \triangleright \tau_{\psi(\hat{j})}((\underline{\theta}_{\hat{j}}^{h^{\text{old}}}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}), \rho^*) \quad (35)$$

Since the new simulated type for $\psi(\hat{j})$ is at least as high as the old simulated type, by Lemma 1 it follows from Equations 34 and 35 that we can pick an alternative history in $\hat{I}_{\hat{j}}$ that is consistent with the new simulated type. That is, since $\theta_{\psi(\hat{j})} \supseteq \theta_{\psi(\hat{j})}^{\text{old}}$, it follows that $\theta_{\psi(\hat{j})} \in \Theta_{\psi(\hat{j})}^{\text{win}}((\underline{\theta}_{\hat{j}}^{h^{\text{old}}}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}), \rho^*)$, so by Lemma 1, there exists $h^{\text{new}} \in \hat{I}_{\hat{j}}$ such that $(\theta_{\psi(\hat{j})}, \underline{\theta}_{\hat{j}}^{h^{\text{old}}}, \underline{\theta}_{N \setminus \{\hat{j}, \psi(\hat{j})\}}^{h^*}) \in \Theta_N^{h^{\text{new}}}$. Thus, we can pick \hat{h} appropriately by defining it to be the immediate successor of h^{new} that is consistent with \hat{j} 's last action, i.e. $\hat{h} := h'' \in \sigma(h^{\text{new}}) \mid \mathcal{A}(h'') = \hat{a}_{\hat{j}}$. Thus, Step 1.d of **First-Query** and steps 2.d and 3.c of **Query** are well-defined.

Observe, finally, that if the algorithm goes from **First-Query** to **Query** to **Conclude**, revenue is at least equal to revenue under S_0^G , and is strictly higher whenever θ_i^* is greater than revenue under S_0^G . Thus, the deviation is safe and results in strictly higher expected profit, and (G, S_N) is not credible. This completes the proof of Lemma 2. \square

Moving from Lemma 2 to Theorem 2 is mostly an exercise in labeling. Bidder i is **active** at h if $\Theta_i^h \cap \Theta_i^{\text{win}}(\underline{\theta}_{-i}, \rho^*) \neq \emptyset$.

There are three cases to consider:

1. An active bidder is called to play, and there is more than one active bidder.
2. An inactive bidder is called to play.
3. An active bidder is called to play, and there are no other active bidders. (We leave this case till last - it can only happen when every other bidder has a type below the reserve.)

Take any I_i and $h \in I_i$ such that an active bidder i is called to play. If there exists another active bidder, then there exists $j \neq i$ such that $\bar{\theta}_j^h \triangleright \tau_i(\underline{\theta}_{-i}^h, \rho^*)$. There is some action $S_i(I_i, \theta_i^K)$ that is taken by the highest type of i . Lemma 2 implies that all the types of i that might still win at h , i.e. $\{\theta_i \mid \theta_i \supseteq \tau_i(\underline{\theta}_{-i}^h, \rho^*)\}$, must also play $S_i(I_i, \theta_i^K)$. Thus, any agent who does not play that action has **quit**. The **bid** at I_i is the least type of i consistent with playing $S_i(I_i, \theta_i^K)$, that is $\min\{\theta_i \in \Theta_i^h \mid S_i(I_i, \theta_i) = S_i(I_i, \theta_i^K)\}$.²⁵

By construction, all types strictly below the bid quit. Since (G, S_N) is d-optimal and has threshold pricing, if there is no high bidder, then all types weakly above ρ^* place a bid. Similarly, all types above the current high bid place a bid.

If bidder i quits, then he either has a type lower than the reserve, or we have identified another bidder whose type is greater than i 's (according to the order \triangleright). Thus, once

²⁵Note that this may be below $\tau_i(\underline{\theta}_{-i}^h, \rho^*)$.

i is inactive, further information about his type no longer affects the outcome, so since (G, S_N) is pruned, only active bidders are called to play. Similarly, if i is the current high bidder at history h , and the auction has not ended, then by Lemma 2, all his types who reach h take the same action, and by (G, S_N) pruned, i is not called to play at h . Thus, if i is called to play at h , he is an active bidder who is not the current high bidder.

Suppose an active bidder i is called to play at h and is the unique active bidder. Since (G, S_N) is pruned, i is not the current high bidder, which implies that there is *no* high bidder - all the other bidders have types below the reserve. Let I_i be such that $h \in I_i$.

In this case, we can define an action a as **quitting** if there is no type above the reserve that plays a , that is:

$$\neg \exists \theta_i \in \Theta_i^h \mid \theta_i \triangleright \rho^* \text{ and } S_i(I_i, \theta_i) = a \quad (36)$$

For any non-quitting action a , the associated **bid** is:

$$\triangleright \min\{\theta_i \mid \theta_i \triangleright \rho^* \text{ or } [\theta_i \in \Theta_i^h \text{ and } S_i(I_i, \theta_i) = a]\} \quad (37)$$

By construction, if i has a type strictly below the bid associated with a , then he does not play a . If i has a type above the reserve, then he places a bid. However, Lemma 2 does not apply (since there are no other active bidders), so there can be multiple actions that place bids.

The three conditions that specify what happens when the auction ends are similarly entailed by d-optimality and threshold pricing. If there are no active bidders at h , then for all i , $\rho^* \triangleright \bar{\theta}_i^h$. Thus, by d-optimality and threshold pricing, the object is not sold, and since (G, S_N) is pruned, h is a terminal history. If the high bidder i is the unique active bidder at h , then we know that no bidder in $N \setminus i$ has a higher type than i , and that $\tau_i(\theta_{-i}, \rho^*)$ is equal to i 's current bid. Thus, by d-optimality and threshold pricing, i must win and pay b , and since (G, S_N) is pruned, h is a terminal history. Finally, if the high bidder has bid θ^K and no active bidder has higher tie-breaking priority, then by d-optimality and threshold pricing, i must win and pay θ^K , and since (G, S_N) is pruned, h is a terminal history.

This completes the proof that if a mechanism is d-optimal, d-strategy-proof, and credible, then it is an ascending auction.

B.6.2 ascending \rightarrow d-strategy-proof, credible

Now we show that if (G, S_N) is d-optimal and an ascending auction, it is d-strategy-proof and credible.

That (G, S_N) is d-strategy-proof is straightforward. (G, S_N) is plainly strategy-proof. Moreover, since (G, S_N) breaks ties in order and is d-optimal, if i is asked at h to place

some required bid b , then b must be no more than

$$\min \theta'_i | \theta'_i \triangleright \max_{j \neq i} \theta_j^h \text{ and } \theta'_i \triangleright \rho^* \quad (38)$$

Moreover, if after i places required bid b at h , the auction then ends, that implies that, for the true type profile θ_N ,

$$\beta = \min \theta'_i | \theta'_i \triangleright \max_{j \neq i} \theta_j \text{ and } \theta'_i \triangleright \rho^* \quad (39)$$

so (G, S_N) has threshold pricing.

It remains to show that (G, S_N) is credible. The first step is to observe that (G, S_N) is not just d-optimal, but *exactly* optimal. In particular, since (G, S_N) has threshold pricing, each type θ_i^k is exactly indifferent between playing as though his type is θ_i^k and playing as though his type is θ_i^{k-1} , since the two strategies only have different outcomes when

$$\theta_i^k = \min \{ \theta'_i | \theta'_i \triangleright \max_{j \neq i} \theta_j \text{ and } \theta'_i \triangleright \rho^* \} \quad (40)$$

in which case both strategies result in zero surplus. Thus, IC binds locally downward. Moreover, F is regular and (G, S_N) maximizes the virtual value of the winning bidder, so by Propositions 3 and 4, (G, S_N) is optimal.

The second step is to observe that for all i , S_i is an *obviously dominant* strategy, in the sense of Li (2017). In particular, for any safe deviation $S'_0 \in \mathcal{S}_0^*(S_0^G)$ and for any S'_{-i} , S_i is a best response to (S'_0, S'_{-i}) in the partial commitment game. We offer a direct proof here:

First, consider information sets at which there is a unique action that places a bid. Take any i , I_i , and θ_i such that $\theta_i \in \Theta_i^{I_i}$. Recall that S_i requires that i quit if θ_i is strictly below the bid $b(I_i)$ at I_i , and that i places the bid if θ_i is above the least high bid consistent with reaching I_i . The least high bid consistent with reaching I_i is, formally,

$$\min \{ \rho^*, \min_{h \in I_i, j \neq i} \theta_j^h \} \quad (41)$$

And, since (G, S_N) is d-optimal and has threshold pricing,

$$b(I_i) \leq \min \{ \theta'_i | \theta'_i \triangleright \min \{ \rho^*, \min_{h \in I_i, j \neq i} \theta_j^h \} \} \quad (42)$$

For any safe deviation S'_0 and for any S'_{-i} , it is optimal for i to quit (upon reaching information set I_i) if $\theta_i \triangleleft \min \{ \rho^*, \min_{h \in I_i, j \neq i} \theta_j^h \}$. In particular, note that under (G, S_N) , if i wins after reaching I_i , he pays at least $\min \{ \rho^*, \min_{h \in I_i, j \neq i} \theta_j^h \}$. Thus, for any safe deviation, i 's best possible payoff upon placing a bid is no more than zero, so it is optimal to quit

(which yields zero payoff).

For any safe deviation S'_0 and for any S'_{-i} , it is optimal for i to place a bid if θ_i is weakly above that bid. This is because i can quit if the bids required ever rise strictly above θ_i . Under any safe deviation, i cannot be charged more than θ_i unless he (at some later point) bids more than θ_i . Thus, the worst possible payoff from placing a bid is zero, and the best possible payoff from quitting is zero.

By the above arguments and Equation 42, there are three possibilities at each I_i and $\theta_i \in \Theta_i^{I_i}$:

1. $\min\{\overset{\triangleright}{\rho^*}, \min_{h \in I_i, j \neq i} \overset{\triangleright}{\theta_j^h}\} \triangleleft \theta_i$, in which case S_i requires that i place a bid, and this is a best response to (S'_0, S'_{-i}) .
2. $\theta_i \triangleleft b(I_i)$, in which case S_i requires i to quit, and this is a best response to (S'_0, S'_{-i}) .
3. $b(I_i) \trianglelefteq \theta_i \triangleleft \min\{\overset{\triangleright}{\rho^*}, \min_{h \in I_i, j \neq i} \overset{\triangleright}{\theta_j^h}\}$, in which case S_i is underdetermined, and both quitting now or placing the bid and quitting later are best responses to (S'_0, S'_{-i}) .

Finally, consider information sets at which there are multiple bid-placing actions. In this case, under any safe deviation, i is sure to win if and only if he eventually bids above the reserve - this implies that his original strategy remains a best response to any safe deviation.

Suppose now that (G, S_N) is not credible, so the auctioneer has a profitable safe deviation S'_0 . Consider a corresponding G' in which the auctioneer ‘pre-commits’ to that deviation, that is to say, G' such that S'_0 runs G' . For all i , S_i is a best response to (S'_0, S_{-i}) , so (G', S_N) is also BIC. (We abuse notation slightly to use S_N as a strategy profile for G and G' . Every information set in G' has a corresponding information set in G , so it is clear what is meant.) By hypothesis, S'_0 is a profitable deviation, so $\pi(G', S_N) > \pi(G, S_N)$. But (G, S_N) is optimal, so this is a contradiction. Thus, if (G, S_N) is d-optimal and an ascending auction, (G, S_N) is credible.